Theoretical and Mathematical Physics, **171**(1): 531–540 (2012)

SPECTRAL DEPENDENCE OF THE LOCALIZATION DEGREE IN THE ONE-DIMENSIONAL DISORDERED LLOYD MODEL

© **G. G. Kozlov***

We calculate the Anderson criterion and the spectral dependence of the degree of localization in the first nonvanishing approximation with respect to disorder for one-dimensional diagonally disordered models with a site energy distribution function that has no finite even moments higher than the zeroth. For this class of models (for which the usual perturbation theory is inapplicable), we show that the perturbation theory can be consistently constructed for the joint statistics of advanced and retarded Green's functions. Calculations for the Lloyd model show that the Anderson criterion in this case is a linear (not quadratic as usual) function of the disorder degree. We illustrate the calculations with computer experiments.

Keywords: Anderson localization, one-dimensional disordered system, Green's function

1. Introduction, formulation of the problem, and basic results

One of the basic reasons for studying one-dimensional ($1D$) systems in solid state physics is the relative simplicity of the corresponding mathematical models. This allows obtaining results for $1D$ models in analytic form in many cases and using these results as guides in studying more realistic two- and three-dimensional models. But there is currently reason to suppose that the value of $1D$ models is not exhausted by this. Contemporary material production technologies and experimental methodologies allow creating and studying low-dimensional systems (e.g., quantum lattices, $1D$ photon crystals, and optical fibers) that may be analyzable using $1D$ mathematical models directly. Moreover, interest in known natural $1D$ systems such as J-aggregates [1], [2], whose spectroscopic properties are described in terms of the $1D$ Frenkel exciton, has recently increased.

Unavoidable variations in the technological processes for producing "synthetic" $1D$ systems and fluctuations of the environment of natural $1D$ systems generate disorder, which must be taken into account in their description. Mathematical models of disorder in $1D$ systems are an important part of disordered system theory [3] and are currently being investigated [4]–[6]. We discuss one class of these models here. The matrix of the Hamiltonian $\mathbf{H}$ for models of this class has the standard form[1]

$$H_{rr'} = \delta_{rr'}\varepsilon_r + \delta_{r,r'+1} + \delta_{r,r'-1}, \quad r, r' = 1, 2, \ldots, N, \quad N \to \infty, \tag{1}$$

and the specific feature is that the distribution function $P(\varepsilon)$ of random site energies ε_r can decrease so slowly that all its even moments except the zeroth moment diverge. The Lloyd model can be an example of such models [7], where $P(\varepsilon)$ is the Cauchy distribution. As is known, the exact calculation of the averaged

*Fock Institute of Physics, St. Petersburg State University, St. Petersburg, Russia,
e-mail: gkozlov@photonics.phys.spbu.ru.

[1]The Hamiltonian of such form describes the Frenkel exciton in the $1D$ diagonally disordered chain with only nearest neighbor interactions.

Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol. 171, No. 1, pp. 124–134, April, 2012. Original article submitted May 19, 2011.

Green's function is possible for the Lloyd model [3], [7], but although the localization of the eigenstate in this model was intensively studied [6], [8]–[12], the results obtained in our work (concerning the spectral dependence of the degree of localization in the sense of the Anderson criterion) have not been explored to the best of our knowledge. We note that the study of such models is interesting methodologically because applying the standard perturbation theory turns out to be impossible in this case.

For analyzing the localization in models of the class described above, we use the methodology in [13], which is based on constructing the joint statistics of advanced and retarded Green's functions. Our goal here is to generalize this methodology to the case of diagonally disordered $1D$ models of the class described above, for which the formulas obtained in [13] expressing the Anderson criterion and the spectral dependence of the localization in terms of the second moment of the site energy distribution function $\int P(\varepsilon)\varepsilon^2\,d\varepsilon$ turn out to be inapplicable. As an example, we present the calculations for the Lloyd model and show that the dependence of the Anderson criterion on the width parameter of the site energy distribution is *linear* in this case, not quadratic as in the case of rapidly decaying distributions [13].

The general formulation of the problem and quantitative definitions of the quantities calculated below correspond exactly to those in [13]. We therefore restrict ourself to a brief review. We consider a long disordered chain described by Hamiltonian (1) and assume that the excitation is concentrated at the edge site N of this chain at the initial instant $t = 0$. We introduce the *Anderson criterion* D as a probability that the excitation remains at the initial site N as $t \to \infty$. As is known, $D = 0$ in the ordered system, and the appearance of localized states (as a consequence of the disorder) in the spectrum of Hamiltonian (1) corresponds to the condition $D \neq 0$ [3], [13], [14].

If $\mathbf{\Psi}^\lambda$ and E_λ are the eigenvectors and eigenvalues of matrix (1), then the quantity D can be represented as [13]

$$D = \int W(U)\,dU, \quad \text{where } W(U)\,dU = \Big\langle \sum_{E_\lambda \in [U, U+dU]} |\Psi_N^\lambda|^4 \Big\rangle. \tag{2}$$

The brackets $\langle\,\cdot\,\rangle$ indicate averaging over the random site energies ε_r. The distinction of the function $W(U)$ from zero at some energy U confirms that the eigenvectors of (1) with the energy U are localized in the sense of the Anderson criterion, and this function can hence be called the *spectral dependence of the degree of localization.*[2]

As in [13], we represent the site energy distribution function $P(\varepsilon)$ as

$$P(\varepsilon) = \frac{1}{\Delta} p\Big(\frac{\varepsilon}{\Delta}\Big), \quad p(x) > 0, \quad \int p(x)\,dx = 1, \tag{3}$$

where Δ is the measure of system disorder because the system becomes ordered as $\Delta \to 0$. The following expressions for D and $W(U)$ were obtained in [13]: $D = M_2\Delta^2/2$ and $W(U) = M_2\Delta^2\sqrt{(4-U^2)}/(4\pi)$, $|U| < 2$, where $M_2 = \int p(x)x^2\,dx$. For $M_2 < \infty$, these expressions are applicable in the case of slightly disordered systems where $D \ll 1$. Our basic results in this paper are the following expressions for the function $W(U)$ and quantity D, which are applicable for $D \ll 1$:

$$W(U) = \frac{(4-U^2)^{3/2}}{4\pi} \int d\varepsilon\, P(\varepsilon) \log\Big(1 + \frac{\varepsilon^2}{4-U^2}\Big), \quad |U| < 2, \quad D = \int_{-2}^{2} dU\, W(U). \tag{4}$$

These expressions are in fact valid for any reasonable function $P(\varepsilon)$. For $M_2 < \infty$, formulas (4) convert (for small Δ) to the formulas in [13]. At $M_2 = \infty$, the dependence of D on Δ changes qualitatively. In particular, for the Lloyd model, where $p(x) = [\pi(1+x^2)]^{-1}$, D turns out to be proportional to the first power of Δ, which is confirmed by the computer experiments described in the appendix.

[2] In [13], $W(U)$ was called a participation function because $W(U)$ has much in common with the inverse participation ratio [4].

2. Calculation

To solve the problem formulated in the preceding section, we use the method proposed in [13], which we now briefly describe. The excitation probability $|\Psi_N(t)|^2$ at the edge site N of the disordered chain is expressed in terms of two edge Green's functions $\gamma(U_1)$ and $\gamma(U_2)$ with different energies U_1 and U_2. These edge Green's functions are random quantities with their joint distribution function $\rho(x_1x_2)$ satisfying the equation [13]

$$x_1^2x_2^2\rho(x_1x_2) = \int P(\varepsilon)\rho\left(U_1 - \frac{1}{x_1} - \varepsilon, U_2 - \frac{1}{x_2} - \varepsilon\right) d\varepsilon. \tag{5}$$

The sought excitation probability $|\Psi_N(t)|^2$ at the edge site is related to the function $\rho(x_1x_2) = \rho_{U_1U_2}(x_1x_2)$ as [13]

$$|\Psi_N(t)|^2 = \frac{i}{\pi^2}\int dU\, d\omega\, e^{-i\omega t}\langle y_2x_1\rangle, \tag{6}$$

where

$$\langle y_2x_1\rangle = \pi \lim_{a\to\infty} a^2 \int x\rho_{U,U+\omega}(x,a)\, dx.$$

For small ω, the function $\rho_{U,U+\omega}(x_1,x_2)$ has a pole-type singularity, i.e., it is representable in the form

$$\rho_{U,U+\omega}(x_1x_2) \approx \operatorname{sing}\rho_{U,U+\omega}(x_1x_2) = \frac{1}{\omega}\mathcal{F}_U(x_1x_2). \tag{7}$$

Substituting representation (7) in (6) and taking the formula $\int d\omega\, e^{-i\omega t}/\omega = -i\pi$ into account, we find that the limit (as $t\to\infty$) of the excitation probability D given by (2) at the edge site is

$$D = |\Psi_N(t=\infty)|^2 = \int W(U)\, dU, \quad \text{where } W(U) = \lim_{a\to\infty} a^2 \int dx\, \mathcal{F}_U(x,a)x, \tag{8}$$

and the function $W(U)$ in this expression, as shown in [13], has the meaning of the spectral dependence of the degree of localization (2). Therefore, to calculate D and $W(U)$, we must find part (7) of the solution of Eq. (5) that is singular with respect to the energy difference ω. This information from [13] is the starting point for the present paper. Below, we describe the scheme for calculating D and $W(U)$ in the first nonvanishing approximation with respect to the disorder degree Δ given by (3), which is applicable for virtually arbitrary reasonable functions $p(x)$ in (3), in particular, in the case where all even moments (except the zeroth moment) of $p(x)$ diverge. As in [13], *all calculations relate to the energy region* $|U| < 2$, where the density of states of the ordered model is nonzero and where the large majority of states are located (which holds in the case of weak disorder that we consider).

We introduce the function $\mathcal{R}(x_1x_2) \equiv \rho(U_1 - x_1, U_2 - x_2)$ for our convenience. Using Eq. (5), we can easily verify that $\mathcal{R}(x_1x_2)$ satisfies the equation

$$\frac{\mathcal{R}(U_1 - 1/x_1, U_2 - 1/x_2)}{x_1^2x_2^2} \equiv \mathcal{H}_{U_1}(x_1)\mathcal{H}_{U_2}(x_2)\mathcal{R}(x_1x_2) = \int P(\varepsilon)\mathcal{R}(x_1+\varepsilon, x_2+\varepsilon)\, d\varepsilon. \tag{9}$$

The functional operator $\mathcal{H}_U(x)$ in the left-hand side of this equation acts on an arbitrary function $f(x)$ as $\mathcal{H}_U(x)f(x) \equiv f(U - 1/x)/x^2$. We use the system of eigenfunctions of $\mathcal{H}_U(x)$ described in [15] to write Eq. (9) in matrix form. We represent the explicit expressions obtained in [15] for the eigenfunctions $\sigma_U^n(x)$ and eigenvalues λ_n of this operator for $|U| < 2$:

$$\sigma_U^n(x) = \mathcal{L}_U(x)\left[\frac{R_U^* - x}{R_U - x}\right]^n \equiv \mathcal{L}_U(x)G_U^n(x), \qquad \lambda_n = \left(\frac{U + i\sqrt{4-U^2}}{U - i\sqrt{4-U^2}}\right)^n, \qquad |\lambda_n| = 1, \tag{10}$$

where

$$G_U(x) \equiv \frac{R_U^* - x}{R_U - x}, \qquad R_U \equiv \frac{U + i\sqrt{4 - U^2}}{2}, \qquad \mathcal{L}_U(x) \equiv \frac{1}{2\pi i}\left[\frac{1}{x - R_U} - \frac{1}{x - R_U^*}\right]. \tag{11}$$

We also need the rule for projecting on the system of functions (10) obtained in [15]: an arbitrary function $f(x)$ can be represented in the series form

$$f(x) = \sum_{n=-\infty}^{+\infty} A_n \sigma_U^n(x), \quad \text{where } A_n = \int \frac{f(x)}{G_U^n(x)}\, dx. \tag{12}$$

Using rule (12), we expand both sides of (9) with respect to the system of functions (10):

$$\begin{aligned} &\mathcal{R}(x_1 x_2) = \sum_{nm} C_{nm} \sigma_{U_1}^n(x_1) \sigma_{U_2}^m(x_2), \\ &\int P(\varepsilon) \mathcal{R}(x_1 + \varepsilon, x_2 + \varepsilon)\, d\varepsilon = \sum_{qp} \sum_{nm} \sigma_{U_1}^q(x_1) \sigma_{U_2}^p(x_2) J_{qp}^{nm} C_{nm}, \end{aligned} \tag{13}$$

where

$$J_{qp}^{nm} \equiv \int P(\varepsilon) \frac{\sigma_{U_1}^n(\varepsilon + x_1) \sigma_{U_2}^m(\varepsilon + x_2)}{G_{U_1}^q(x_1) G_{U_2}^p(x_2)}\, dx_1\, dx_2\, d\varepsilon.$$

Equating the coefficients of $\sigma_{U_1}^n(x_1)\sigma_{U_2}^m(x_2)$ in the two sides of (9), we obtain

$$C_{qp} \lambda_q(U_1) \lambda_p(U_2) = \sum_{n,m} J_{qp}^{nm} C_{nm}. \tag{14}$$

We label the pairs of indices with a single index $qp \to \alpha$, where the index pair $q = p = 0$ is associated with $\alpha = 1$. Setting $\lambda_q(U_1)\lambda_p(U_2) \equiv \Lambda_\alpha$, we can then write (14) for all α as

$$\sum_\beta M_{\alpha\beta} C_\beta = 0, \qquad M_{\alpha\beta} = J_\alpha^\beta - \Lambda_\alpha \delta_{\alpha\beta}. \tag{15}$$

Equation (15) is the sought matrix equivalent of Eq. (9).

The solution of Eq. (5) has the meaning of a distribution function and must therefore have a fixed sign. The arguments confirming that *any* solution of (5) has a fixed sign are given in the appendix. We show that the proportionality of all solutions (5) to each other (i.e., the one-dimensionality of the solution manifold of (5)) hence follows. Indeed, we suppose that there are two linearly independent solutions of Eq. (5), ρ_1 and ρ_2. They can be considered positive functions without loss of generality. An arbitrary linear combination $\rho = \rho_1 c_1 + \rho_2 c_2$ is then a solution, the coefficients c_1 and c_2 can be taken with different signs, and the function ρ then changes its sign at some values of the arguments x_1 and x_2. This contradicts the property that the solutions of (5) have a fixed sign, which is established in the appendix. Consequently, there cannot be *two* (*or more*) *linearly independent solutions of Eq.* (5) (and of Eqs. (9) and (15), which are related to it). For matrix equation (15), this means that all rows of a degenerate matrix M (i.e., $\det M = 0$) are linearly independent except *a single* row.

Let the row of matrix equation (15) corresponding to $\alpha = 1$ be a linearly dependent row. Because $\int \sigma_U^n(x)\, dx = \delta_{n0}$ [15], the normalization condition $\int dx_1\, dx_2\, \mathcal{R}(x_1 x_2) = 1$ corresponds to the requirement that $C_{00} = 1 \to C_1 = 1$. If we now let L denote the size of the matrix M (of course, we should take $L \to \infty$ in the calculations), then we can easily show that the normalized solution of Eq. (15) can be written as

$$\begin{aligned} &C_\alpha = K(\Delta) e_{\alpha, i_2, i_3, \ldots, i_L} M_{2 i_2} M_{3 i_3} M_{4 i_4} \ldots M_{L i_L}, \quad L \to \infty, \\ &K(\Delta) = [e_{1, i_2, i_3, \ldots, i_L} M_{2 i_2} M_{3 i_3} M_{4 i_4} \ldots M_{L i_L}]^{-1}, \quad L \to \infty, \end{aligned} \tag{16}$$

where summation from 1 to L is assumed for the repeated indices and $e_{iklm\ldots n}$ is the Levi-Civita symbol (the totally antisymmetric function of indices).[3] We recall that, first, $e_{123\ldots L} = 1$ and, second, the transposition of any two indices leads to a sign change (e.g., $e_{21345\ldots L} = -1$). Consequently, $e_{iklm\ldots n} \neq 0$ only if there are no coinciding indices among $iklm\ldots n$ (e.g., $e_{11345\ldots L} = 0$).

We assume that we are dealing with the ordered system, i.e., $\Delta = 0$ and $P(\varepsilon) = \delta(\varepsilon)$. Then $J_\alpha^\beta(\Delta = 0) = \delta_{\alpha\beta}$ and consequently $M_{\alpha\beta}(\Delta = 0) = \delta_{\alpha\beta}(1 - \Lambda_\alpha)$. From formula (16), we obtain $C_\alpha(\Delta = 0) = \delta_{\alpha 1}$ (this result coincides with the result in [13]) and

$$K(0) = \prod_{\alpha=1}^{L} (1 - \Lambda_\alpha)^{-1}, \quad L \to \infty. \tag{17}$$

In the transition to the disordered system, nondiagonal elements appear in the matrix M. They are small with respect to the disorder Δ in the sense that they vanish as $\Delta \to 0$. Therefore, to obtain the first nonvanishing approximation of the coefficients C_α in the products of the matrix elements in formula (16), we keep only one of the nondiagonal elements and regard the other (diagonal) elements as unperturbed. The normalization coefficient $K(\Delta)$ must then be equal to $K(0)$ in (17) in the calculation of the first nonvanishing correction. In this approach, the type of smallness (integer degree, fractional degree, or any other) of the nondiagonal elements of the matrix M as functions of Δ plays no role. Thus, for the coefficients C_α in the first nonvanishing approximation (shown by the approximate equalities), we obtain

$$\begin{aligned} C_\alpha \approx K(0)[&e_{\alpha,i_2,3,4,5,\ldots,L} J_2^{i_2}(1 - \Lambda_3)(1 - \Lambda_4)\cdots(1 - \Lambda_L) + \\ &+ e_{\alpha,2,i_3,4,5,\ldots,L}(1 - \Lambda_2) J_3^{i_3}(1 - \Lambda_4)\cdots(1 - \Lambda_L) + \\ &+ e_{\alpha,2,3,i_4,5,\ldots,L}(1 - \Lambda_2)(1 - \Lambda_3) J_4^{i_4}(1 - \Lambda_5)\cdots(1 - \Lambda_L) + \ldots]. \end{aligned} \tag{18}$$

In the first term in the square brackets, the nondiagonal element is in the position M_{2i_2} (see formula (16)). Taking the explicit expression for $K(0)$ given by (17) into account, we can easily see that this term can be transformed as

$$K(0) e_{\alpha,i_2,3,4,5,\ldots,L} J_2^{i_2}(1 - \Lambda_3)(1 - \Lambda_4)\cdots(1 - \Lambda_L) = e_{\alpha,i_2,3,4,5,\ldots,L} \frac{J_2^{i_2}}{1 - \Lambda_2}.$$

This expression is nonzero only if $\alpha = 1$ or $\alpha = 2$. The case $\alpha = 1$ corresponds to calculating the correction to the normalization constant and is uninteresting for us [13]. In the case $\alpha = 2$ and $i_2 = 1$, the considered term yields the coefficient C_2 because the other terms in expression (18) are zero at $\alpha = 2$. Analogous results can be obtained for all $\alpha \neq 1$:

$$C_\alpha \approx -\frac{J_\alpha^1}{1 - \Lambda_\alpha}, \quad \alpha \neq 1. \tag{19}$$

As previously noted, we are interested only in terms in expansion (13) that are singular with respect to $\omega = U_2 - U_1$. This singularity occurs only in coefficients (19) such that the index α corresponds to a pair of indices with opposite signs, i.e., $\alpha \to n, -n$ [13]. The singularity appears as a zero of the denominator in (19) at $\omega = 0$, and the energy arguments in all quantities (except the denominator in (19)) can therefore

[3] For example, if $L = 3$, then two linearly independent rows of the matrix M can be associated with two nonparallel vectors $\mathbf{a}$ and $\mathbf{b}$. Solving (15) then means finding a vector $\mathbf{c}$ that is orthogonal to those two vectors. This vector is parallel to the vector product $[\mathbf{a}, \mathbf{b}]$, whose components are known to be written using the three-dimensional Levi-Civita tensor: $c_i \sim e_{ijk} a_j b_k$.

be equalized by setting $U_1 = U_2 \equiv U$. Hence, the quantities J^1_α are defined by the integrals

$$J^1_\alpha \to J^{00}_{-nn} = \int P(\varepsilon) \frac{\mathcal{L}_U(\varepsilon + x_1)\mathcal{L}_U(\varepsilon + x_2)}{G^n_U(x_1)G^{-n}_U(x_2)}\, dx_1\, dx_2\, d\varepsilon =$$

$$= \int d\varepsilon\, P(\varepsilon) \left(\frac{\varepsilon^2}{\varepsilon^2 + 4 - U^2} \right)^{|n|}. \tag{20}$$

The integrals over x_1 and x_2 in expression (20) were calculated in [13]. Therefore, for the singular part of the first nonvanishing correction to the function $\mathcal{R}(x_1x_2)$ with respect to the disorder (denoted by $\operatorname{sing}\mathcal{R}(x_1x_2)$), we can write the expressions

$$\operatorname{sing}\mathcal{R}(x_1x_2) = \sum_{n\neq 0} C_{-nn}\sigma^n_U(x_1)\sigma^{-n}_U(x_2) =$$

$$= -\sum_{n\neq 0} \frac{J^{00}_{-nn}}{1 - \lambda_n(U_1)\lambda_{-n}(U_2)} \sigma^n_U(x_1)\sigma^{-n}_U(x_2). \tag{21}$$

The expansion (see Eq. (10))

$$1 - \lambda_n(U)\lambda_{-n}(U + \omega) = -\frac{2in\omega}{\sqrt{4 - U^2}} + O(\omega^2) \tag{22}$$

holds for small ω, and therefore

$$\operatorname{sing} R(x_1x_2) = \frac{\sqrt{4 - U^2}}{2i\omega} \sum_{n\neq 0} J^{00}_{-nn} \frac{\sigma^n_U(x_1)\sigma^{-n}_U(x_2)}{n}, \tag{23}$$

whence, using the properties of functions (10), $\sigma^n_U(U - x) = \sigma^{-n}_U(x)$, we obtain the expression for the function $\mathcal{F}_U(x_1x_2)$ in Eq. (7):

$$\mathcal{F}_U(x_1x_2) = \frac{\sqrt{4 - U^2}}{2i} \sum_{n\neq 0} J^{00}_{-nn} \frac{\sigma^{-n}_U(x_1)\sigma^n_U(x_2)}{n}. \tag{24}$$

Using Eqs. (8), (20), and (24), we can easily obtain

$$W(U) = -\frac{i}{2}\sqrt{4 - U^2} \sum_{n\neq 0} \int d\varepsilon\, \frac{P(\varepsilon)}{n} \left(\frac{\varepsilon^2}{\varepsilon^2 + 4 - U^2} \right)^{|n|} \times$$

$$\times \int \sigma^n_U(x)x\, dx \lim_{a\to\infty} \sigma^{-n}_U(a)a^2. \tag{25}$$

Using the properties of functions (10) (proved in [13])

$$\int \sigma^n_U(x)x\, dx = \frac{i}{2}\frac{n}{|n|}\sqrt{4 - U^2}, \qquad \lim_{a\to\infty} \sigma^{-n}_U(a)a^2 = \frac{\sqrt{4 - U^2}}{2\pi}, \tag{26}$$

and the fact that the series in expression (25) can be readily summed using the formula $\sum_{n=1}^{\infty} q^n/n = -\log(1 - q)$, we obtain expressions (4) for the function $W(U)$ and quantity D.

As an example, we consider the Lloyd model [7] with $p(x) = [\pi(1+x^2)]^{-1}$. Calculating the integral in (4) using the residues, we can show that for this model,

$$W(U) = \frac{[4-U^2]^{3/2}}{2\pi} \log\left(1 + \frac{\Delta}{\sqrt{4-U^2}}\right).$$

In the case of small Δ that is important for us, we obtain $W(U) \approx (4-U^2)\Delta/(2\pi)$ and $D \approx 16\Delta/(3\pi)$. We recall that these quantities are always proportional to Δ^2 for the models with a finite second moment M_2 considered in [13]. The dependence of the Anderson criterion D on the disorder Δ and the spectral dependence of the degree of localization $W(U)\,dU|_{\Delta=0.007}$ obtained for the Lloyd model in the computer experiment are shown in Fig. 1.[4] The theoretical dependences shown there were obtained using formulas (4) without any adjustment. They allow estimating the validity of the calculations in the present paper. It can be seen in Fig. 1a that for small disorder, the Anderson criterion D for the Lloyd model depends linearly on the disorder parameter Δ.

3. Conclusion

We have calculated the value of the Anderson criterion and the spectral dependence of the degree of localization for the disordered $1D$ models whose distribution function $P(\varepsilon)$ of the site energy does not have finite even moments other than the zeroth moment. For calculations, we generalized the method for constructing the joint statistics of the advanced and retarded Green's functions (proposed in [13]) to the case of random systems with divergent even moments of the function $P(\varepsilon)$. The suggested calculation scheme is based on the system of special functions [15] and seems more compact and universal than that described in [13]. It allows calculating the Anderson criterion and the spectral dependence of the degree of localization in cases where the usual perturbation theory is inapplicable. As an example, we considered the Lloyd model [7] and obtained a nontrivial linear dependence of the Anderson criterion on the degree of disorder.

Appendix

We argue that any solution of Eq. (5) with $|U_{1,2}| < 2$, at least in the case of weak disorder, is a function of fixed sign. It immediately follows from (5) that as $x_1, x_2 \to \pm\infty$, the function $\rho(x_1x_2) \approx x_1^{-2}x_2^{-2}\int d\varepsilon\, P(\varepsilon)\rho(U_1-\varepsilon, U_2-\varepsilon)$ and consequently the function $\rho(x_1x_2)$ has a fixed sign at large absolute values of x_1 and x_2. We ensure that the function $\rho(x_1x_2)$ has a fixed sign in the absence of disorder. Equation (5) in this case has the form

$$x_1^2x_2^2\rho(x_1x_2) = \rho\left(U_1 - \frac{1}{x_1}, U_2 - \frac{1}{x_2}\right). \tag{27}$$

It hence follows that the function $\rho(x_{1,n}, x_{2,n})$ preserves its sign in the manifold of points with coordinates $(x_{1,n}, x_{2,n})$ such that $x_{i,n+1} = U_i - 1/x_{i,n}$, $i = 1, 2$ (we choose the starting point $(x_{1,1}, x_{2,1})$ in the region of large absolute values of x_1 and x_2, where the sign is preserved in accordance with the above remarks). The process of constructing the set of points x_n for each variable can be clearly represented in the standard manner using graphs of the functions $y = U - 1/x$ and $y = x$. The first few steps of this process are shown in Fig. 2. As is known, the points x_n obtained by the successive application of the map $x_{n+1} = U - 1/x_n$ for $|U| < 2$ (where the equation $x = U - 1/x$ does not have a real roots) fill the number axis with the density defined by the Cauchy distribution [15], which is nonzero everywhere. Therefore, as $n \to \infty$, the

[4] The methodology of the computer experiment is described in [13].

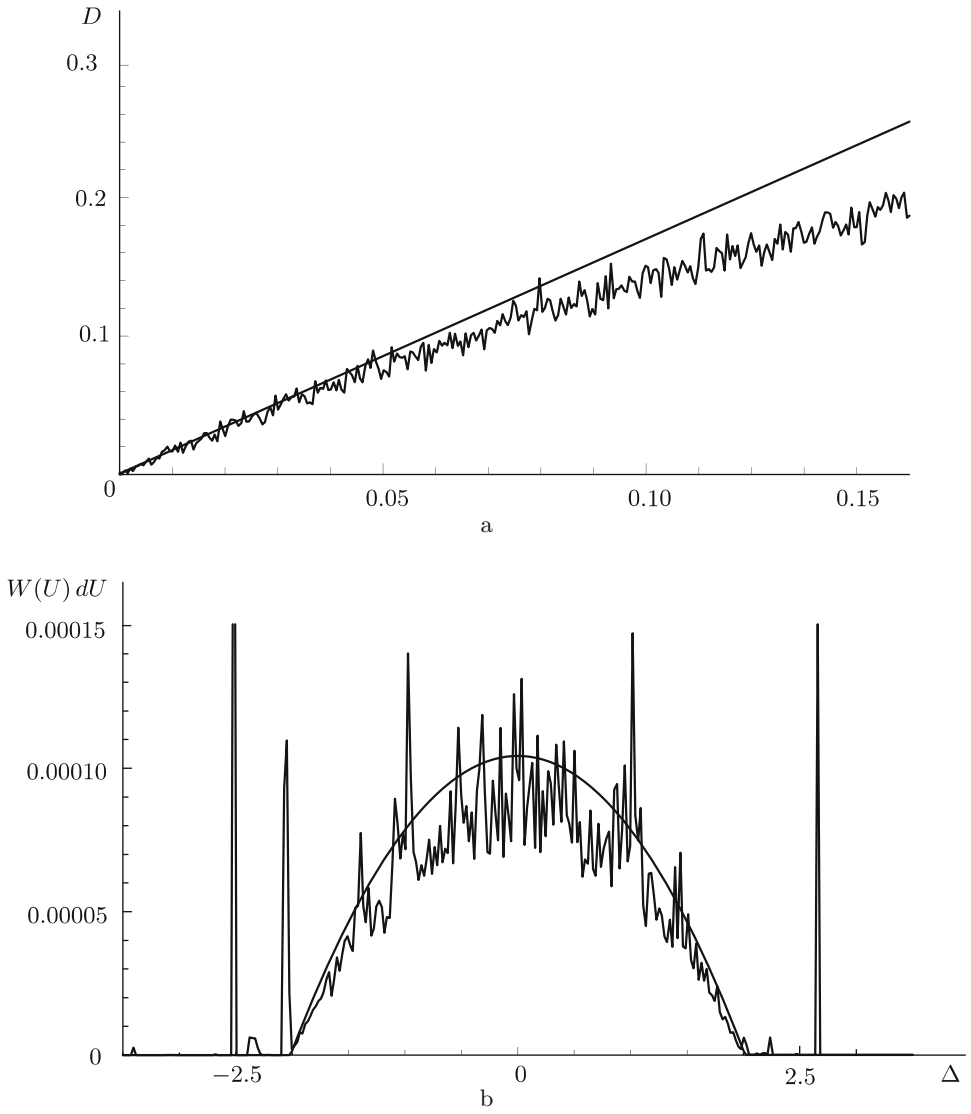

Fig. 1. (a) Dependence of the Anderson criterion D on the disorder parameter Δ for the Lloyd model: the smooth line corresponds to calculating by formulas (4), and the noisy line corresponds to the computer experiment. (b) Distribution of the degree of localization $W(U)dU$ for the $1D$ Lloyd model for $\Delta = 0.007$ and $dU = 7/300$: the smooth curve corresponds to calculating by formula (4), and the noisy curve corresponds to the computer experiment with averaging over 2000 realizations of random matrices (1) of size $N = 2000$.

points $(x_{1,n}, x_{2,n})$ (where the function $\rho(x_1x_2)$ has the same sign) "cover" the whole plane (x_1, x_2), and the function $\rho(x_1x_2)$ consequently has a fixed sign in the absence of disorder. The arbitrary choice of the starting point gives an additional reason for such a conclusion.

Analogous reasoning can be used in the case of small disorder. If the distribution function $P(\varepsilon)$ of the site energy is nonzero only over some finite interval $[-a, a]$ (i.e., $\int_{-a}^{a} P(\varepsilon)\, d\varepsilon = 1$), then the relation

$$\int P(\varepsilon) f(x - \varepsilon)\, d\varepsilon = f(x - \bar{\varepsilon}(x)) \tag{28}$$

holds for an arbitrary function $f(x)$, where the function $\bar{\varepsilon}(x)$ satisfies the condition $\bar{\varepsilon}(x) \in [-a, a]$ for any

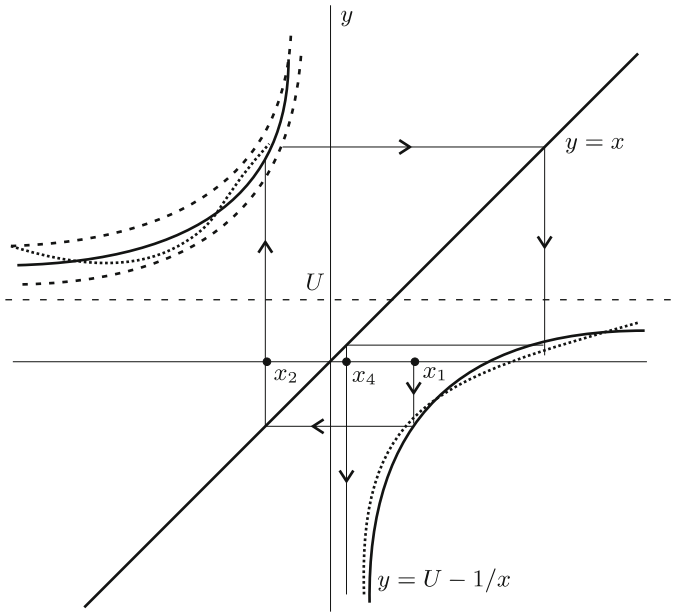

Fig. 2. Constructing the sequence of points x_n.

x. If the function $P(\varepsilon)$ is nonzero everywhere, then relation (28) also holds, but we can only say that the function $\bar{\varepsilon}(x)$ is bounded and tends to zero as the disorder decreases ((i.e., as $\Delta \to 0$ in (3)). Keeping this in mind, we rewrite Eq. (5) as

$$x_1^2 x_2^2 \rho(x_1 x_2) = \rho\left(U_1 - \frac{1}{x_1} - \bar{\varepsilon}, U_2 - \frac{1}{x_2} - \bar{\varepsilon}\right) \equiv \rho\left(\widetilde{U}_1 - \frac{1}{x_1}, \widetilde{U}_2 - \frac{1}{x_2}\right), \tag{29}$$

where $\bar{\varepsilon}(x_1 x_2)$ is a bounded function, $\varepsilon_{\min} < \bar{\varepsilon}(x_1 x_2) < \varepsilon_{\max}$, which tends to zero as $\Delta \to 0$. As can be seen, Eq. (29) differs from Eq. (27) only by replacing $U_{1,2}$ with $\widetilde{U}_{1,2} \equiv U_{1,2} - \bar{\varepsilon}(x_1 x_2)$, where the function $\bar{\varepsilon}(x_1 x_2)$ is small for small disorder. We now propose a graphic construction of the system of points $(x_{1,n}, x_{2,n})$ (where the function $\rho(x_1 x_2)$ preserves the sign) analogous to that described above for the case of zero disorder. The hyperbolic function $U - 1/x$ must then be replaced with $U - \bar{\varepsilon}(x, z) - 1/x$, and constructing $x_{1,n+1}$, we must set $x = x_{1,n}$, $z = x_{2,n}$, while constructing $x_{2,n+1}$, we set $x = x_{2,n}$, $z = x_{1,n}$. This substitution is schematically shown in Fig. 2 by the dashed line. For small disorder, the above substitution may not qualitatively change the picture of constructing the manifold of points where the function ρ has the same sign because for the limit functions $U_{1,2} - \varepsilon_{\min} - 1/x$ and $U_{1,2} - \varepsilon_{\max} - 1/x$ (which are shown in Fig. 2 by the dashed lines in the second quadrant), the construction does not differ from that described above for $\Delta = 0$. A qualitative change might be possible, for instance, if there are intersection points of the graphs $y = \widetilde{U}(x) - 1/x$ and $y = x$. But for small disorder (and consequently for a small function $\bar{\varepsilon}(x_1 x_2)$), this possibility can occur only if the energy parameters $|U_{1,2}|$ are in the small region in the vicinity of 2, whose size vanishes as $\Delta \to 0$. This is inessential for the calculations of the first nonvanishing approximation to the spectral dependence of the localization $W(U)$, which are performed in this paper.

REFERENCES

1. E. E. Jelley, *Nature*, **38**, 1009–1010 (1936).
2. G. Scheibe, *Angew. Chem.*, **50**, No. 11, 212–219 (1937).
3. I. M. Lifshits, S. A. Gredeskul, and L. A. Pastur, *Introduction to the Theory of Disordered Systems* [in Russian], Nauka, Moscow (1982).

4. A. Rodríguez, V. A. Malyshev, G. Sierra, M. A. Martín-Delgado, J. Rodríguez-Laguna, and F. Domínguez-Adame, *Phys. Rev. Lett.*, **90**, 027404 (2003).
5. V. A. Malyshev, A. Rodrígues, and F. Domíngues-Adame, *Phys. Rev. B*, **60**, 14140–14146 (1999); arXiv:cond-mat/9908438v1 (1999).
6. M. Titov and H. Schomerus, *Phys. Rev. Lett.*, **91**, 176601 (2003); arXiv:cond-mat/0302580v1 (2003).
7. P. Lloyd, *J. Phys. C*, **2**, 1717–1725 (1969).
8. D. J. Thouless, *J. Phys. C*, **16**, L929 (1983).
9. A. MacKinnon, *J. Phys. C*, **17**, L289–L292 (1984).
10. D. E. Rodrigues, H. M. Pastawski, and J. F. Weisz, *Phys. Rev. B*, **34**, 8545–8549 (1986).
11. S. Fishman, R. E. Prange, and M. Griniasty, *Phys. Rev. A*, **39**, 1628–1633 (1989).
12. L. I. Deych, A. A. Lisyansky, and B. L. Altshuler, *Phys. Rev. Lett.*, **84**, 2678–2681 (2000).
13. G. G. Kozlov, *Theor. Math. Phys.*, **162**, 238–253 (2010).
14. P. W. Anderson, *Phys. Rev.*, **109**, 1492–1505 (1958).
15. G. G. Kozlov, "Spectrum and eigen functions of the operator $H_U f(x) \equiv f(U - 1/x)/x^2$ and strange attractor's density for the mapping $x_{n+1} = 1/(U - x_n)$," arXiv:0803.1920v1 [math-ph] (2008).