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CALCULATION OF THE SPECTRAL DEPENDENCE OF THE ANDERSON LOCALIZATION CRITERION IN A ONE-DIMENSIONAL SYSTEM WITH CORRELATED DIAGONAL DISORDER

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We consider the problem of calculating the Anderson criterion for a one-dimensional disordered chain with correlated disorder. We solve this problem by the perturbation method with the inverse correlation length as the small parameter. We show that in a correlated system, the degree of localization not only naturally decreases but its spectral dependence also differs significantly from the spectral dependence in uncorrelated chains. The calculations are based on the method for constructing joint statistics of Green's functions, which was previously used to analyze uncorrelated one-dimensional systems. We illustrate the theoretical calculations with a numerical experiment.

Keywords: Anderson localization, correlated disorder, Green's function

1. Introduction, statement of the problem, and the main results

The most popular mathematical models of spatially homogeneous disordered systems are models where the disorder is represented by a delta-correlated random field. These are the simplest models, and one possible step increasing their complexity is to introduce correlations. The problem of determining the influence of correlations is very relevant because it is natural to expect the appearance of a correlated disorder in real systems. Moreover, as shown in [1]–[3], the most important property of random systems (localization of eigenstates) can significantly depend on the presence of correlations.

To describe the localization in uncorrelated low-dimensional random systems, it is expedient to use the widely known notion of localization length [4]. The same can be said about correlated systems [5]–[8]. Despite a high informational potential of the notion of localization length, using such notions does not always permit obtaining a comprehensive conclusion about the character (localized–delocalized) of eigenstates of a random system [9], [10] (also see [5]). An additional study of localization in random systems (both correlated and uncorrelated) using the basic Anderson criterion [4], [11] therefore seems necessary.

Here, we analyze the localization of eigenstates in a one-dimensional correlated chain using the method proposed by Dyson [12]. The development of this method allowed obtaining several results about the spectral [13], [14] and localization [15]–[18] properties of one-dimensional disordered systems. The cited papers deal with uncorrelated random chains; the only known exception is [16], where a random chain with a complicated structure unit was analyzed. This chain can be considered a correlated chain (but rather artificially).

We calculate the spectral dependence of the degree of localization in the sense of the Anderson criterion for the simplest diagonally disordered correlated one-dimensional chain and demonstrate that there is a significant difference between this dependence and the dependence in the case of an uncorrelated chain.

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We use the method proposed in [15]–[17] for our calculations. The method is based on constructing joint statistics of advanced and retarded Green's functions. The small parameter in the calculations given below is the inverse correlation length, and the obtained formulas work for a significant disorder value (with respect to the unperturbed Hamiltonian) in cases where the usual quantum mechanical perturbation theory cannot be applied. Therefore, the obtained results once again demonstrate the effectiveness and universality of the method of joint statistics of Green's functions proposed in [15]–[17], which permits analyzing a class of problems of which some (e.g., the Lloyd problem [17]) are quantum mechanically nonperturbative.

We now pose the problem. The matrix of the Hamiltonian of a disordered chain of length N considered in this paper has the standard form [4] in the site representation (the case of diagonal disorder)

$$H_{n,n'} = \delta_{n,n'}\varepsilon_n + \delta_{n,n'+1} + \delta_{n,n'-1}, \quad n, n' = 1, \ldots, N, \tag{1}$$

but in contrast to the traditional statement of the problem, the random site energies ε_n now form a correlated sequence with the following simplest correlation mechanism, similar to that described, for example, in [5]. We associate an auxiliary random variable ξ_n with each site n of our chain and assume that the variables ξ_n, $n = 1, \ldots, N$, are mutually independent with a prescribed distribution density $P(x)$. We use these variables to construct the site energies ε_n as

$$\varepsilon_n = (1 - e^{-\alpha}) \sum_{m \le n} e^{\alpha(m-n)}\xi_m, \quad \alpha > 0.$$

The constructed variables ε_n, $n = 1, \ldots, N$, are correlated, and the correlation function $\langle \varepsilon_n \varepsilon_{n'} \rangle$ has the exponential form

$$\langle \varepsilon_n \varepsilon_{n'} \rangle = \langle \xi^2 \rangle \left(\frac{1 - e^{-\alpha}}{1 + e^{-\alpha}} \right) e^{-\alpha|n-n'|} = \langle \xi^2 \rangle \left(\frac{1-\beta}{1+\beta} \right) \beta^{|n-n'|}, \qquad \beta \equiv e^{-\alpha}, \quad \beta \in [0, 1],$$

with the characteristic correlation length $R = 1/\alpha$. We note that the sequence ε_n, $n = 1, \ldots, N$, is causally correlated, i.e., the random variable ε_n depends only on ξ_m for $m \le n$. The key point in our further considerations is the relation

$$\varepsilon_{n+1} = \beta\varepsilon_n + (1 - \beta)\xi_{n+1}, \tag{2}$$

which can be verified by straightforward calculations.

In what follows, we assume that the distribution density of the auxiliary variables ξ_n is Gaussian,

$$P(x) = \frac{1}{\Delta} \frac{1}{\sqrt{\pi}} e^{-(x/\Delta)^2}. \tag{3}$$

We can easily see that the distribution density $\mathcal{P}(z)$ of the site energies ε_n is then also Gaussian,

$$\mathcal{P}(z) = \frac{1}{\sqrt{\pi}} \frac{1}{\bar{\Delta}} e^{-(z/\bar{\Delta})^2}, \quad \text{where } \bar{\Delta} \equiv \Delta \sqrt{\frac{1-\beta}{1+\beta}}. \tag{4}$$

For such a chain with a correlated diagonal disorder, we consider the standard problems of calculating the Anderson criterion D [4], [15]–[17], i.e., of calculating the residual density of excitation at the edge site of the chain (see [18] for the relation between the parameter D and the localization region), and determining the spectral dependence of the localization degree $W(U)$ [15]. These quantities can be expressed in terms of the eigenvalues E_λ and the boundary components of the eigenvectors $\mathbf{\Psi}^\lambda$ of matrix (1) as

$$W(U)\, dU = \left\langle \sum_{E_\lambda \in [U, U+dU]} |\Psi_N^\lambda|^4 \right\rangle, \qquad D = \int W(U)\, dU. \tag{5}$$

It follows from the methods proposed in [15]–[17] that to solve the above problem, we must know the joint statistic of the two edge Green's functions (EGF) $g_N(\Omega_1)$ and $g_N(\Omega_2)$ for different (complex) energies $\Omega_1 = U_1 + iV_1$ and $\Omega_2 = U_2 - iV_2$, where $V_{1,2} > 0$. The rules for calculating the Anderson criterion D and the function $W(U)$ using this statistic are described in [15]–[17]. In what follows, we obtain the corresponding formulas with the correlated character of the site energies taken into account. We immediately note that in contrast to the problems analyzed in [15]–[17], we can obtain a closed equation in the considered case of correlated disorder only for the joint statistic of two edge Green's functions $g_N(\Omega_{1,2})$ and the edge site energy ε_N. This statistic is a function $\rho(z_1, z_2, \varepsilon)$ of two complex arguments and one real argument and is defined such that the quantity $\rho(x_1 + iy_1, x_2 + iy_2, \varepsilon)\, dx_1\, dy_1\, dx_2\, dy_2\, d\varepsilon$ is equal to the probability of the joint occurrence of the events

$$\begin{aligned} &\operatorname{Re} g_N(\Omega_1) \in [x_1, x_1 + dx_1], \qquad \operatorname{Im} g_N(\Omega_1) \in [y_1, y_1 + dy_1], \\ &\operatorname{Re} g_N(\Omega_2) \in [x_2, x_2 + dx_2], \qquad \operatorname{Im} g_N(\Omega_2) \in [y_2, y_2 + dy_2], \end{aligned} \qquad \varepsilon_N \in [\varepsilon, \varepsilon + d\varepsilon].$$

Our first problem is to derive a closed equation for the function $\rho(z_1, z_2, \varepsilon)$. For this, just as in [15]–[17], we use a known relation (see, e.g., [4]) that permits expressing the EGF $g_{N+1}(\Omega)$ of a chain with one added site in terms of the EGF $g_N(\Omega)$ of the initial chain as

$$g_{N+1}(\Omega) = \frac{1}{\Omega - \varepsilon_{N+1} - g_N(\Omega)}. \tag{6}$$

To make the calculations more compact, it is convenient to use the following definition of the delta function in which the argument is an analytic function $F(z) = u(x, y) + iv(x, y)$ of the complex variable $z = x + iy$:

$$\delta[F(z)] \equiv \delta[u(x, y)]\delta[v(x, y)]. \tag{7}$$

Such a delta function is always real and can be transformed as follows. Let $z_0 = x_0 + iy_0$ be a root of the function $F(z)$. Straightforward calculations with the Cauchy–Riemann relations for the function $F(z)$ taken into account then show that

$$\delta[F(z)] = \frac{\delta[z - z_0]}{|F'(z_0)|^2}, \qquad F(z_0) = 0, \tag{8}$$

where the delta function in the numerator of the complex function of $z - z_0$ is understood in the sense of (7).

For brevity, we also let $\int_R dz$ to denote the integrals of the (not necessarily analytic) functions $f(z) = \tilde{u}(x, y) + i\tilde{v}(x, y)$ of the complex variable $z = x + iy$:

$$\int_R f(z)\, dz \equiv \iint_{-\infty}^{+\infty} [\tilde{u}(x, y) + i\tilde{v}(x, y)]\, dx\, dy.$$

We now return to deriving the equation for the joint statistic $\rho(z_1, z_2, \varepsilon)$ and add one more site to our chain that already contains N sites: $N \to N + 1$. If we let $\tilde{\rho}(z_1, z_2, \varepsilon)$ denote a function that is similar to ρ but only describes the joint statistic of the EGF of the chain with the added site, then we can write the

expression for it with the above definitions and relations (2) and (6) taken into account:

$$\begin{aligned}\tilde\rho(z_1,z_2,\varepsilon)=&\left\langle\delta\biggl(z_1-\frac{1}{\Omega_1-\varepsilon_{N+1}-g_N(\Omega_1)}\biggr)\delta\biggl(z_2-\frac{1}{\Omega_2-\varepsilon_{N+1}-g_N(\Omega_2)}\biggr)\delta(\varepsilon-\varepsilon_{N+1})\right\rangle=\\ =&\left\langle\delta\biggl(z_1-\frac{1}{\Omega_1-\varepsilon-g_N(\Omega_1)}\biggr)\delta\biggl(z_2-\frac{1}{\Omega_2-\varepsilon-g_N(\Omega_2)}\biggr)\delta(\varepsilon-\beta\varepsilon_N-(1-\beta)\xi_{N+1})\right\rangle=\\ =&\int_R dz_1'\,dz_2'\int P(x)\rho(z_1'z_2',\varepsilon')\times\\ &\times\delta\biggl(z_1-\frac{1}{\Omega_1-\varepsilon-z_1'}\biggr)\delta\biggl(z_2-\frac{1}{\Omega_2-\varepsilon-z_2'}\biggr)\delta(\varepsilon-\beta\varepsilon'-(1-\beta)x)\,d\varepsilon'\,dx.\end{aligned}$$

As usual, the relation $\rho=\tilde\rho$ (the condition that the statistic is stationary) must hold as $N\to\infty$ (the thermodynamic limit). We use formula (8) to integrate with delta functions and obtain a closed equation for the stationary joint distribution function $\rho(z_1,z_2,\varepsilon)$:

$$\beta|z_1|^4|z_2|^4\rho(z_1,z_2,\varepsilon)=\int P(x)\rho\biggl(\Omega_1-\varepsilon-\frac{1}{z_1},\Omega_2-\varepsilon-\frac{1}{z_2},\frac{\varepsilon-(1-\beta)x}{\beta}\biggr)dx.$$

To calculate the residual density D and the function $W(U)$, we need Green's functions such that the imaginary parts of their energy arguments are as small as possible and have opposite signs. By analogy with [15]–[17], this fact can be used to reduce the problem to the analysis the equation for the joint statistic of the real EGF and the edge-site energy:

$$\beta x_1^2x_2^2\sigma(x_1,x_2,\varepsilon)=\int P(x)\sigma\biggl(U_1-\varepsilon-\frac{1}{x_1},U_2-\varepsilon-\frac{1}{x_2},\frac{\varepsilon-(1-\beta)x}{\beta}\biggr)dx. \tag{9}$$

The further calculations are also based on a direct generalization of the results and formulas in [15]–[17], where it was shown that to calculate the function $W(U)$, it is necessary to know only the part of the function $\sigma(x_1,x_2,\varepsilon)$ that is singular in the difference $U_2-U_1\equiv\omega$ between the energy arguments and that this singularity has the form of a pole. We thus obtain $\sigma(x_1,x_2,\varepsilon)\approx\operatorname{sing}\sigma(x_1,x_2,\varepsilon)$ for small ω, where $\operatorname{sing}\sigma(x_1,x_2,\varepsilon)$ can be written as

$$\operatorname{sing}\sigma(x_1,x_2,\varepsilon)=\frac{1}{\omega}\mathcal{F}_U(x_1,x_2,\varepsilon). \tag{10}$$

We here change the notation as $U_1\to U$. If the function $\mathcal{F}_U(x_1,x_2,\varepsilon)$ is known, then the function $W(U)$ and the Anderson criterion D can be calculated by the formulas

$$W(U)=\lim_{a\to\infty}a^2\int\mathcal{F}_U(x,a,\varepsilon)\,dx\,d\varepsilon,\qquad D=\int W(U)\,dU. \tag{11}$$

The derivation of formulas (9)–(11) is completely similar to that of the corresponding formulas in [15]–[17].

The problem thus reduces to the determination of the singular part $\mathcal{F}(x_1,x_2,\varepsilon)/\omega$ of the solution of Eq. (9). In the next section, to solve this problem, we develop a perturbation theory in which the small parameter is the inverse correlation length $1/R=-\log\beta$ and obtain expressions for the spectral dependence of the localization degree $W(U)$ and the Anderson criterion D:

$$\begin{aligned}W(U)&=\frac{\bar\Delta^2}{4\pi R}\operatorname{Re}\int\frac{\mathcal{P}(\varepsilon)}{\sqrt{4-(U-\varepsilon)^2}}\,d\varepsilon+O\biggl(\frac{1}{R^2}\biggr),\\ D&=\int W(U)\,dU=\frac{\bar\Delta^2}{4R}+O\biggl(\frac{1}{R^2}\biggr).\end{aligned} \tag{12}$$

These expressions can be applied to the correlated diagonally disordered chain with large correlation lengths $R\gg1$. We note that these formulas can also work in the case $\bar\Delta>1$, i.e., for significant values of disorder.

2. Approximation of large correlation lengths

Let the correlation length satisfy the condition $R \gg 1$, and hence let $\beta \approx 1$. To obtain an approximate solution of Eq. (9), we construct a perturbation theory in a small parameter $y \equiv 1 - \beta > 0$ ($y \approx 1/R$). We assume that the distribution density of site energies has form (4) and $\bar{\Delta}$ is fixed and not necessarily small. This means that the width Δ of distribution (3) of the auxiliary variables ξ_n increases as $\beta \to 1$. The distribution density $P(x)$ of the auxiliary variables ξ_n thus depends on β and has form (3), where

$$\Delta = \bar{\Delta}\sqrt{\frac{1+\beta}{1-\beta}}. \tag{13}$$

For the further reasoning, it is convenient to pass to a new function $\mathcal{R}(x_1, x_2, \varepsilon)$ in Eq. (9) using the formula

$$\mathcal{R}(U_1 - \varepsilon - x_1, U_2 - \varepsilon - x_2, \beta\varepsilon) \equiv \sigma(x_1, x_2, \varepsilon). \tag{14}$$

Equation (9) for the function $\mathcal{R}(x_1, x_2, \varepsilon)$ then implies

$$\beta\frac{\mathcal{R}(U_1 - \varepsilon - 1/x_1, U_2 - \varepsilon - 1/x_2, \beta\varepsilon)}{x_1^2 x_2^2} = \int P(x)\mathcal{R}(x_1, x_2, \varepsilon - yx)\, dx. \tag{15}$$

We consider the integral in the right-hand side of this equation. The function $P(x)$ is significantly different from zero for $x \in [-\Delta, \Delta]$, and the dimension of the corresponding range of the variable yx in (15) can be estimated as

$$\Delta y = \bar{\Delta}\sqrt{1-\beta^2} \approx \bar{\Delta}\sqrt{2y} \to 0 \quad \text{as } \beta \to 1.$$

For small y, the function $\mathcal{R}(x_1, x_2, \varepsilon - yx)$ in the right-hand side of (15) can therefore be expanded in a power series up to the second term (the first term is absent because the first moment of the function $P(x)$ is zero):

$$\beta\frac{\mathcal{R}(U_1 - \varepsilon - 1/x_1, U_2 - \varepsilon - 1/x_2, \beta\varepsilon)}{x_1^2 x_2^2} = \mathcal{R}(x_1, x_2, \varepsilon) + \frac{\partial^2\mathcal{R}(x_1, x_2, \varepsilon)}{\partial\varepsilon^2}\frac{y^2\Delta^2}{4}.$$

We here used the fact that $\int x^2 P(x)\, dx = \Delta^2/2$ for function (3). We substitute Δ given by (13), take the relation $\beta \approx 1$ into account, and obtain

$$\beta\frac{\mathcal{R}(U_1 - \varepsilon - 1/x_1, U_2 - \varepsilon - 1/x_2, \beta\varepsilon)}{x_1^2 x_2^2} = \mathcal{R}(x_1, x_2, \varepsilon) + \frac{\partial^2\mathcal{R}(x_1, x_2, \varepsilon)}{\partial\varepsilon^2}\frac{y\bar{\Delta}^2}{2}. \tag{16}$$

We now recall that the function $\mathcal{R}$ directly depends on y. In the linear approximation, this dependence has the form

$$\mathcal{R}(x_1, x_2, \varepsilon) = \mathcal{R}_0(x_1, x_2, \varepsilon) + y\mathcal{R}_1(x_1, x_2, \varepsilon) + \ldots.$$

We substitute this expression in (16) and equate the coefficients of like powers of y. We equate the coefficients of y^0 and obtain

$$\frac{\mathcal{R}_0(U_1 - \varepsilon - 1/x_1, U_2 - \varepsilon - 1/x_2, \varepsilon)}{x_1^2 x_2^2} = \mathcal{R}_0(x_1, x_2, \varepsilon). \tag{17}$$

Taking this equation into account and equating the coefficients of y, we obtain

$$[\mathcal{H}_{U_1-\varepsilon}(x_1)\mathcal{H}_{U_2-\varepsilon}(x_2) - 1]\mathcal{R}_1(x_1, x_2, \varepsilon) = \mathcal{A}, \tag{18}$$

where

$$\mathcal{A} \equiv \frac{\bar{\Delta}^2}{2}\frac{\partial^2\mathcal{R}_0(x_1, x_2, \varepsilon)}{\partial\varepsilon^2} + \frac{\varepsilon}{x_1^2 x_2^2}\frac{\partial\mathcal{R}_0(U_1 - \varepsilon - 1/x_1, U_2 - \varepsilon - 1/x_2, \theta)}{\partial\theta}\bigg|_{\theta=\varepsilon} + \mathcal{R}_0(x_1, x_2, \varepsilon). \tag{19}$$

In Eq. (18), we let $\mathcal{A}$ denote the right-hand side of the equation for $\mathcal{R}_1$ and introduce the functional operator $\mathcal{H}_U(x)$ [15], [19] acting on an arbitrary function $f(x)$ by the rule $\mathcal{H}_U(x)f(x) \equiv f(U-1/x)/x^2$. It is known [15], [19] that the eigenfunction of this operator corresponding to the unit eigenvalue is the Lorentzian $\mathcal{L}_U(x)$, whose explicit expression is given below (see formulas (24)). With this in mind, it is easy to see that the solution $\mathcal{R}_0$ of Eq. (17) has the form

$$\mathcal{R}_0(x_1, x_2, \varepsilon) = \mathcal{L}_{U_1-\varepsilon}(x_1)\mathcal{L}_{U_2-\varepsilon}(x_2)\mathcal{P}(\varepsilon), \tag{20}$$

where $\mathcal{P}(\varepsilon)$ is the distribution density of site energies in form (4).

To solve functional equation (18) for the first nonvanishing correction $\mathcal{R}_1$, we expand the right- and left-hand sides of this equation in the eigenfunctions $\sigma^n_{U_1-\varepsilon}(x_1)\sigma^m_{U_2-\varepsilon}(x_2)$ of $\mathcal{H}_{U_1-\varepsilon}(x_1)\mathcal{H}_{U_2-\varepsilon}(x_2)$, as was done in [15]–[17], and equate the corresponding coefficients. The expansion for $\mathcal{R}_1$ is

$$\mathcal{R}_1(x_1, x_2, \varepsilon) = \sum_{m,n} C_{nm}\sigma^n_{U_1-\varepsilon}(x_1)\sigma^m_{U_2-\varepsilon}(x_2). \tag{21}$$

To obtain the coefficients C_{nm}, we must expand the known right-hand side of Eq. (18) in the functions $\sigma^n_{U_1-\varepsilon}(x_1)\sigma^m_{U_2-\varepsilon}(x_2)$ and equate the corresponding coefficients. For this, we recall the explicit expressions for the eigenfunctions $\sigma^n_U(x)$ and the eigenvalues λ_n of $\mathcal{H}_U(x)$ for $|U| < 2$, which were obtained in [19]:

$$\sigma^n_U(x) = \mathcal{L}_U(x)\left(\frac{R^*_U - x}{R_U - x}\right)^n \equiv \mathcal{L}_U(x)G^n_U(x), \tag{22}$$

$$\lambda_n = \left(\frac{U + i\sqrt{4-U^2}}{U - i\sqrt{4-U^2}}\right)^n, \qquad |\lambda_n| = 1, \tag{23}$$

where

$$\begin{aligned} &G_U(x) \equiv \frac{R^*_U - x}{R_U - x}, \qquad R_U \equiv \frac{U + i\sqrt{4-U^2}}{2}, \\ &\mathcal{L}_U(x) \equiv \frac{1}{2\pi i}\left(\frac{1}{x - R_U} - \frac{1}{x - R^*_U}\right). \end{aligned} \tag{24}$$

We also recall the rule for projecting on the system of functions (22), which states that an arbitrary function $f(x)$ can be represented as a series

$$f(x) = \sum_{n=-\infty}^{+\infty} A_n\sigma^n_U(x), \quad \text{where } A_n = \int \frac{f(x)}{G^n_U(x)}dx.$$

It follows from Eq. (18) that $\int \mathcal{A}\, dx_1\, dx_2 = 0$. This implies that the function $\mathcal{P}(\varepsilon)$ given by (4) must satisfy the equation

$$\frac{\bar{\Delta}^2}{2}\frac{d^2\mathcal{P}}{d\varepsilon^2} + \varepsilon\frac{d\mathcal{P}}{d\varepsilon} + \mathcal{P}(\varepsilon) = 0, \tag{25}$$

which we obtain from the right-hand side of Eq. (18) after substituting $\mathcal{R}_0$ in form (20) and integrating over x_1 and x_2. A straightforward verification shows that Eq. (25) is indeed satisfied.

With the foregoing in mind, we obtain the expressions for the coefficients C_{nm} (except for the coefficient C_{00}, which is determined by normalization and is not singular) in formula (21):

$$C_{nm} = \frac{A_{nm}}{\lambda_n(U_1-\varepsilon)\lambda_m(U_2-\varepsilon) - 1}, \qquad A_{nm} = \int \frac{\mathcal{A}(x_1, x_2)}{G^n_{U_1-\varepsilon}(x_1)G^m_{U_2-\varepsilon}(x_2)}\, dx_1\, dx_2. \tag{26}$$

We now recall that we need only the part of the operator $\mathcal{R}_1$ that is singular in $\omega = U_2 - U_1$ and is associated with the terms of the expansion with $n = -m$ [15], namely, the denominator vanishes for $U_1 = U_2$ precisely in coefficients (26) of these terms. For small ω, we have the expansion

$$1 - \lambda_n(U_1 - \varepsilon)\lambda_{-n}(U_1 + \omega - \varepsilon) = -\frac{2in\omega}{\sqrt{4 - (U_1 - \varepsilon)^2}} + O(\omega^2).$$

Only the first term in expression (19) for $\mathcal{A}$ contributes to the singular part of interest to us, and we calculate for only this term.[1] We introduce the notation $a_n \equiv A_{n,-n}$ for the corresponding coefficients of the expansion. At this stage of the calculations, we can already equate the energy arguments, $U_1 = U_2 \equiv U$, because the singularity is determined only by the denominator of coefficients (26) with $n = -m$. We have

$$a_n = \frac{\bar{\Delta}^2}{2}\frac{d^2}{d\theta^2}\left(\int \frac{\mathcal{R}_0(x_1, x_2, \theta)}{G^n_{U-\varepsilon}(x_1)G^{-n}_{U-\varepsilon}(x_2)}\, dx_1\, dx_2\right)\Big|_{\theta=\varepsilon}.$$

As in [15]–[17], we determine the quantities $J_n(\theta)$ as

$$J_n(\theta) \equiv \int \frac{\mathcal{L}_{U-\theta}(x)}{G^n_{U-\varepsilon}(x)}dx, \qquad J_{-n}(\theta) = J^*_n(\theta), \qquad J_n(\varepsilon) = 0 \quad \text{for } n \neq 0.$$

Applying expression (23) and integrating with the residue theorem, we can show that

$$J_n(\theta) = \left(\frac{R_{U-\varepsilon} - R_{U-\theta}}{R^*_{U-\varepsilon} - R_{U-\theta}}\right)^n, \quad n > 0. \tag{27}$$

We then have

$$a_n = \frac{\bar{\Delta}^2}{2}\frac{d^2\big(|J_n(\theta)|^2\mathcal{P}(\theta)\big)}{d\theta^2}\Big|_{\theta=\varepsilon}, \tag{28}$$

and we obtain an expression of form (10) for the singular part[2] of the function $\sigma(x_1, x_2, \varepsilon)$ (denoted by "sing")

$$\operatorname{sing}\sigma(x_1, x_2, \varepsilon) = \frac{1}{\omega}\mathcal{F}_U(x_1, x_2, \varepsilon), \tag{29}$$

where

$$\mathcal{F}_U(x_1, x_2, \varepsilon) = -\frac{iy}{2}\sqrt{4 - (U - \varepsilon)^2}\sum_{n\neq 0}\frac{a_n}{n}\sigma^n_{U-\varepsilon}(U - \varepsilon - x_1)\sigma^{-n}_{U-\varepsilon}(U - \varepsilon - x_2) \tag{30}$$

and the quantities a_n are determined by formula (28). We can neglect the difference between β and unity in the functions σ and in the coefficients $A_{n,-n} = a_n$ (they depend on ε, and it is necessary hence to take their values at the point $\beta\varepsilon$ in the transition from $\mathcal{R}(x_1, x_2, \varepsilon)$ to $\sigma(x_1, x_2, \varepsilon)$ by formula (14)). Straightforward calculations show that only the terms with $n = \pm 1$ in sum (30) are nonzero. We use relations (28) and (29) to obtain the expression for $a_1 = a_{-1}$:

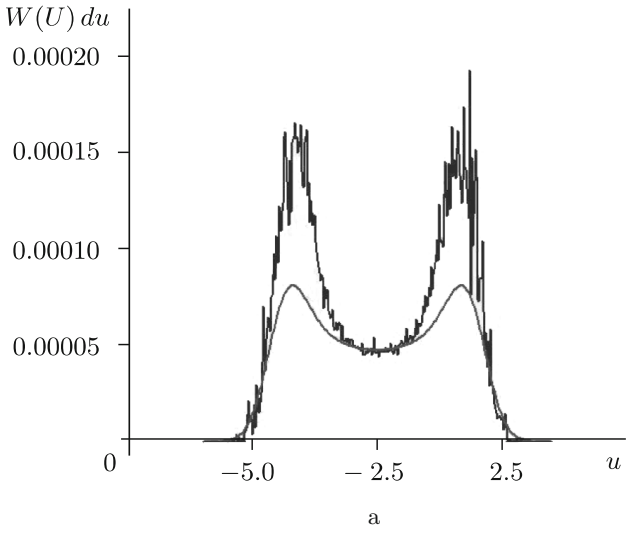
$$a_1 = \mathcal{P}(\varepsilon)\frac{\bar{\Delta}^2}{[4 - (U - \varepsilon)^2]^2}.$$

We use the properties of functions (22) proved in [15],

$$\int \sigma^n_U(x)x\, dx = \frac{i}{2}\frac{n}{|n|}\sqrt{4 - U^2}, \qquad \lim_{a\to\infty}\sigma^{-n}_U(a)a^2 = \frac{\sqrt{4 - U^2}}{2\pi},$$

[1] Following the calculation scheme given below, we can see that the other terms in (19) do not contribute.

[2] This part is related to the singular part of the function $\mathcal{R}$ by change of variables (14).



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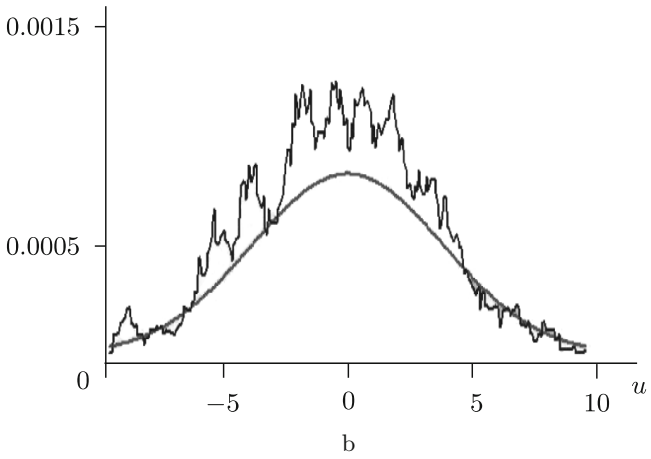

b

Fig. 1. Spectral dependence of the degree of localization in one-dimensional correlated chains for different values of the degree of disorder and the correlation lengths: (a) for $\bar{\Delta} = 0.5$ and $R = 5$ with 3000 realizations and (b) for $\bar{\Delta} = 5$ and $R = 50$ with 200 realizations. The random matrix dimension was 2000 in all cases. The noisy graphs correspond to the computer experiment, and the smooth graphs were obtained by formula (12) without any fitting. As the correlation length increases, the consistency between the theory and experiments improves although the degree of disorder increases.

recall that $y = 1/R$ (where R is the correlation length, which we assume to be rather large), apply formulas (11), and obtain expressions (12) for $W(U)$ and the residual density D.

A numerical verification of formulas (12) using relations (5) is somewhat hindered by the fact that for large correlation lengths R, the dimensions of random matrices (1) must be taken approximately R times greater than those required to verify similar formulas in [15], [17]. The results of the numerical verification are shown in Fig. 1. Figure 1a was obtained for the parameter values $\bar{\Delta} = 0.5$ and $R = 5$. Because the value of the correlation length is small, Fig. 1a demonstrates only the qualitative correspondence between the theory and the numerical experiment. Figure 1b, which was obtained for $\bar{\Delta} = 5$ and $R = 50$, shows that an increase in the correlation length improves the consistency between the theoretical and numerical dependences because formulas (12) were obtained in the approximation of large correlation lengths.

3. Conclusion

We have demonstrated that the method for analyzing the localization in one-dimensional systems, proposed in [15]–[17] and based on constructing the joint statistics of advanced and retarded Green's functions, can be generalized to the case of correlated low-dimensional disordered systems. In the framework of this method, it is possible to construct a simple perturbation theory for the statistics of Green's functions, where the small parameter is the inverse correlation length $1/R$, and to show that the residual density D at the edge site (the Anderson criterion) becomes R times lower compared with its value obtained for an uncorrelated chain [15]. This result seems to be natural, i.e., the initially created excitation at the edge site necessarily spreads over a region whose dimension is of the order of the correlation length.

The second result obtained here seems more unexpected. This result means that the spectral dependence $W(U)$ of the localization degree given by (12) differs significantly from the spectral dependence obtained for uncorrelated chains [15] and is a smoothed density of states of an ordered chain.

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