# Optically trapped polariton condensates as semiclassical time crystals 

A. V. Nalitov, ${ }^{1,2}$ H. Sigurdsson, ${ }^{1}$ S. Morina, ${ }^{1}$ Y. S. Krivosenko, ${ }^{2}$ I. V. Iorsh, ${ }^{2}$ Y. G. Rubo, ${ }^{3}$ A. V. Kavokin, ${ }^{4,5}$ and I. A. Shelykh ${ }^{1,2}$<br>${ }^{1}$ Science Institute, University of Iceland, Dunhagi 3, IS-107, Reykjavik, Iceland<br>${ }^{2}$ ITMO University, St. Petersburg 197101, Russia<br>${ }^{3}$ Instituto de Energías Renovables, Universidad Nacional Autónoma de México, Temixco, Morelos 62580, Mexico<br>${ }^{4}$ Westlake University, 18 Shilongshan Road, Hangzhou 310024, Zhejiang Province, China<br>${ }^{5}$ Institute of Natural Sciences, Westlake Institute for Advanced Study, 18 Shilongshan Road, Hangzhou 310024, Zhejiang Province, China

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#### Abstract

We analyze nonequilibrium phase transitions in microcavity polariton condensates trapped in optically induced annular potentials. We develop an analytic model for annular optical traps, which gives an intuitive interpretation for recent experimental observations on the polariton spatial mode switching with variation of the trap size. In the vicinity of polariton lasing threshold we then develop a nonlinear mean-field model accounting for interactions and gain saturation, and identify several bifurcation scenarios leading to formation of high angular momentum quantum vortices. For experimentally relevant parameters we predict the emergence of spatially and temporally ordered polariton condensates (time crystals), which can be witnessed by frequency combs in the polariton lasing spectrum or by direct time-resolved optical emission measurements. In contrast to previous realizations, our polaritonic time crystal is spontaneously formed from an incoherent excitonic bath and does not inherit its frequency from any periodic driving field.


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## I. INTRODUCTION

The idea of a time crystal, a state of matter characterized with discrete translation symmetry in both space and time, was recently proposed by Wilczek [1,2], shortly followed by the establishment that the absence of thermodynamic equilibrium is a necessary prerequisite for its realization [3-5]. Recently, two groups reported observation of time crystals in periodically driven discrete trapped ion systems [6] and dipolar interacting diamond impurities [7]. Physical systems that support time crystal behavior include Bose-Einstein condensates of atoms [8], magnons [9], and photonic cavities [10] (see Ref. [11] for review). The observation of time translation symmetry breaking in time crystals extends the limits of relativistic analogy between space and time and thus has a huge fundamental significance.

The absence of thermodynamic equilibrium is naturally fulfilled for exciton-polariton condensates in microcavities, created by continuous incoherent optical or electric pumping [12], as the typical polariton thermalization time is longer than its lifetime. Although the Mermin-Wagner theorem forbids two-dimensional bosonic condensation with long-range order [13], trapped cavity polaritons may macroscopically populate a size quantized single-particle state [14]. Among different polariton trapping schemes, such as mechanical strain [14] or cavity etching [15], optically created traps have recently attracted significant attention due to extremely high tunability [16-18]. The optical confining potential stems from polariton repulsion off an inhomogeneous excitonic reservoir, typically generated by a spatially modulated light beam. At the same time, the reservoir provides an inflow of polaritons to compensate for their decay, mainly governed by photon escape from the cavity, and, above a certain threshold density, supports
a stationary condensate population $[19,20]$. Nonequilibrium polariton condensates may thus occupy an excited trapped single-particle mode should it have higher net gain than the ground state.

Polariton-polariton interaction plays a crucial role here, as it can lift occasional degeneracy of the state, occupied by the driven condensate, and spontaneously break either translational [21], spatial inversion [22], or parity symmetry [23]. Moreover, polariton-polariton interaction, supplemented with dissipative coupling, is sufficient for condensate stabilization in the weak lasing regime [22]. Finally, parametric polaritonpolariton scattering out of the condensate may populate several energy levels, resulting in a multimode condensation [24].

Polaritons repel through electron or hole Coulomb exchange interaction due to their excitonic component, governed by the Hopfield coefficient [25,26]. However, in the presence of a hot excitonic reservoir, condensed polaritons may also effectively attract due to local reservoir depletion [27] and lattice heating [28], resulting in condensate instability through self-localization [29]. Destabilized resonantly driven polariton condensates may follow limit cycles and chaotic dynamics in both their density [30] and polarization [31,32]. The limit cycle behavior is also inherent to the weak lasing regime [33].

In this work we focus on the properties of a single annular optical trap, where a condensate occupies a quasidegenerate excited mode doublet. The natural basis of modes in a rotational symmetric Hamiltonian are then the angular harmonics. In such optical traps the formation of pinned and stable quantum vortices was recently predicted [34] and observed $[18,35]$. At the same time, polariton condensation into spatially ordered high angular momentum modes was observed in wider optical traps [36,37]. We demonstrate the interrelation between these observations and point out that a
spatially and temporally ordered phase has been missed so far.

We derive the analytic model of optically trapped polariton condensates in Sec. II, starting with the linear approximation in Sec. II A and extending it above the polariton condensation threshold in Sec. II B. We then address the case of spatially and temporally ordered phase of the condensate in Sec. III, where this phase is studied both within the two-mode model in Sec. III A and by a direct numerical simulation of the mean-field model in Sec. III B. We finally discuss the obtained results and review the conclusions in Sec. IV.

## II. ANALYTIC MODEL

We describe the condensate with the two-dimensional $\left(\nabla^{2}=\partial_{x}^{2}+\partial_{y}^{2}\right)$ complex Gross-Pitaevskii equation (GPE),

$$
\begin{equation*}
i \frac{\partial \Psi}{\partial t}=\left[-\frac{\nabla^{2}}{2 m_{p}}+\frac{N}{2}(\alpha+i \beta)+\frac{\alpha_{1}}{2}|\Psi|^{2}-i \frac{\Gamma}{2}\right] \Psi \tag{1}
\end{equation*}
$$

coupled to the semiclassical Boltzmann equation describing a reservoir of excitons sustaining the condensate,

$$
\begin{equation*}
\frac{\partial N}{\partial t}=\mathrm{P}(\mathbf{r})-\left(\beta|\Psi|^{2}+\gamma\right) N \tag{2}
\end{equation*}
$$

Here $\hbar=1, \Psi$, and $N$ are the condensate order parameter and the reservoir density, $m_{p}$ is the effective lower polariton mass, $\alpha$ and $\alpha_{1}$ are the interaction constants describing the polariton repulsion off the exciton density and polariton-polariton repulsion, $\beta$ governs the stimulated scattering from the reservoir into the condensate, $\Gamma$ and $\gamma$ are the polariton and exciton decay rates, and $\mathrm{P}(\mathbf{r})$ is the inhomogeneous reservoir pumping rate [38]. Assuming that the hot excitonic reservoir dynamics are fast on the time scale of the condensate evolution, its stationary density obtained from Eq. (2) enters Eq. (1) as a function of $|\Psi|^{2}$, supplementing it with additional nonlinear terms [39].

Small optical traps of sizes comparable to the characteristic reservoir variation scale are well approximated with a harmonic trapping potential [40], although its rotational symmetry is normally broken [41]. A potential created by an elliptic paraboloid reservoir density profile lifts the degeneracy of motion along the two principal axes, which explains the formation of "ripple" polariton modes in small optical traps [37]. To describe wider traps we rather employ a rotationally symmetric box trap by assuming a sharp edge between the reservoir $r>R$, and reservoir free region $r<R$, where $r$ is the radial coordinate of the planar microcavity system. The former then corresponds to a uniform real potential and gain region provided by the homogeneous reservoir of density $N$. Although realistic traps have an outer radius, we neglect polariton tunneling out of the trap governed by evanescent tails of confined state wave function into the barrier, keeping in mind that this approximation fails in the vicinity of confined state transitions to the continuum.

## A. Linear limit

In the linear regime $\left(|\Psi|^{2} \simeq 0\right)$ and for a stationary reservoir $(d N / d t=0)$, solutions to Eq. (1) can be written in the form $\Psi^{\mathrm{n}, \mathrm{m}}(r, \varphi)=\exp (i \mathrm{~m} \varphi) \Psi^{\mathrm{n}, \mathrm{m}}(r)$, with m and n being the
angular and the radial quantum numbers respectively. The radial part $\Psi^{\mathrm{n}, \mathrm{m}}(r)$ can be found in the two regions and has a piecewise defined form

$$
\Psi^{\mathrm{n}, \mathrm{~m}}(r)=\left\{\begin{array}{ll}
\left.A^{\mathrm{n}, \mathrm{~m}} J_{\mathrm{m}}\left(r \sqrt{2 m_{p}\left(E^{\mathrm{n}, \mathrm{~m}}+\mathrm{i} \Gamma / 2\right.}\right)\right), & r<R  \tag{3}\\
B^{\mathrm{n}, \mathrm{~m}} K_{\mathrm{m}}\left(r \sqrt{2 m_{p}\left(U-E^{\mathrm{n}, \mathrm{~m}}\right)}\right), & r>R
\end{array},\right.
$$

where $J_{\mathrm{m}}$ and $K_{\mathrm{m}}$ are the analytic continuations of the Bessel function of the first kind and the Macdonald function of the second kind respectively, with their arguments being unambiguously defined as the principal square-root values, and $U=(\alpha+i \beta) N / 2-i \Gamma / 2$. Here the complex energies $E^{\mathrm{n}, \mathrm{m}}$, as well as the normalization constants $A^{\mathrm{n}, \mathrm{m}}$ and $B^{\mathrm{n}, \mathrm{m}}$, are defined from equating the values and the first derivatives of the two wave-function parts at the trap edge $r=R$, which yields the complex transcendental equation

$$
\begin{equation*}
-s_{2} J_{\mathrm{m}}\left(s_{1}\right) K_{\mathrm{m}-1}^{\prime}\left(s_{2}\right)=s_{1} K_{\mathrm{m}}\left(s_{2}\right) J_{\mathrm{m}}^{\prime}\left(s_{1}\right) \tag{4}
\end{equation*}
$$

where the prime denotes differentiation, $s_{1}=$ $R \sqrt{2 m_{p}\left(E^{\mathrm{n}, \mathrm{m}}+i \Gamma / 2\right)}$, and $s_{2}=R \sqrt{2 m_{p}\left(U-E^{\mathrm{n}, \mathrm{m}}\right)}$.

A polariton lasing threshold is defined as a point of gainloss equilibrium of the linearized GPE (1), given by the condition $\operatorname{Im}\left\{E^{\mathrm{n}, \mathrm{m}}\right\}=0$, which occurs at a certain threshold reservoir density $N=N_{t}^{\mathrm{n}, \mathrm{m}}$ in the region $r>R$. Among the modes with a given angular momentum m the ground radial state $\mathrm{n}=0$ has the lowest threshold, which explains why only this type of mode is observed at the threshold in large traps. The physical reason behind this is the centrifugal force pushing a rotating condensate into the gain region. The dimensionless gain in the barrier $p_{t}^{0, \mathrm{~m}}=\beta N_{t}^{0, \mathrm{~m}} / \Gamma$, obtained by numerical solution of Eq. (4), is plotted in Fig. 1(a) for the experimentally relevant relation $\alpha / \beta=5$ as a function of the dimensionless trap radius $\rho=R \sqrt{2 m_{p} \Gamma}$. Every mode m has a critical trap radius of transition to the continuum, where $E_{t}^{0, \mathrm{~m}}=\alpha N_{t}^{0, \mathrm{~m}} / 2$. With increasing trap size $\rho$ the angular momentum of the polariton lasing threshold mode consequently increases through a cascade of successive switchings, as shown in Fig. 1(b). The angular momentum switching behavior, as well as the superlinear oscillating pumping threshold dependence on the trap size, qualitatively reproduces the experimental data in Ref. [37].


FIG. 1. Results of the linear non-Hermitian rectangular trap model. (a) Reservoir pumping rate at the lasing threshold for states with $\mathrm{m}=0,1, \ldots, 8$ as a function of the trap size $\rho=R \sqrt{2 m_{p} \Gamma}$. Condensate switching points are marked with circles and dashed lines. (b) Condensate angular quantum number $m$ at the polariton lasing threshold as a function of the only two linear model parameters.

Any disorder or asymmetry in the pump profile will lift the degeneracy of the system and pin the orientation of the modes. The angular dependence of the lasing threshold mode density is therefore $|\Psi(\varphi)|^{2} \propto 1+\cos (2 m \varphi)$, as shown in the insets of Fig. 1(a).

## B. Two-mode model

The role of the nonlinearity of Eq. (1) becomes increasingly important for pumping powers above the lasing threshold. Assuming small trap asymmetry we neglect all modes except for the doublet $\pm \mathrm{m}$, corresponding to the condensation threshold. We must also introduce the corresponding angular harmonics of the reservoir density $N(\varphi)=n_{0}+n_{1} \cos (2 \varphi)+$ $n_{2} \sin (2 \varphi)$ in the pumped region to account for the its coupling to the condensate. Projecting Eq. (1) onto the basis $\Psi^{0, \pm \mathrm{m}}$ ( $\Psi=\psi_{+} \Psi^{0, \mathrm{~m}}+\psi_{-} \Psi^{0,-\mathrm{m}}$ ) we have in the rotating wave frame:

$$
\begin{align*}
i \frac{d \psi_{ \pm}}{d t}= & \frac{1}{2}(\alpha+i \beta) I_{c r}\left[n_{0} \psi_{ \pm}+\frac{n_{1} \mp i n_{2}}{2} \psi_{\mp}\right] \\
& +\frac{1}{2} \alpha_{1} I_{c c}\left[\left|\psi_{ \pm}\right|^{2}+2\left|\psi_{\mp}\right|^{2}\right] \psi_{ \pm} \tag{5}
\end{align*}
$$

where $I_{c r}=2 \pi \int_{R}^{+\infty}\left|\Psi^{0, \mathrm{~m}}\right|^{4} r d r$ is the condensate wavefunction overlap with the resevoir, $I_{c c}=2 \pi \int_{0}^{+\infty}\left|\Psi^{0, \mathrm{~m}}\right|^{4} r d r$ is the effective condensate overlap with itself, and the reservoir density angular harmonics, defined from the condition on the stationary reservoir density $d N / d t=0$, being

$$
\begin{align*}
& n_{0}=\left(P-I_{c r} p_{t}^{0, \mathrm{~m}} \Gamma s\right) / \gamma \\
& n_{1}=\left(\delta P-I_{c r} p_{t}^{0, \mathrm{~m}} \Gamma s_{x}\right) / \gamma,  \tag{6}\\
& n_{2}=-I_{c r} p_{t}^{0, \mathrm{~m}} \Gamma s_{y} / \gamma,
\end{align*}
$$

and pseudospin components describing the condensate state defined as

$$
\begin{align*}
s_{x} & =\operatorname{Re}\left\{\psi_{+}^{*} \psi_{-}\right\}, \\
s_{y} & =\operatorname{Im}\left\{\psi_{+}^{*} \psi_{-}\right\},  \tag{7}\\
s_{z} & =\left(\left|\psi_{+}\right|^{2}-\left|\psi_{-}\right|^{2}\right) / 2
\end{align*}
$$

$P$ is the angle independent part of the reservoir pumping rate variation from its threshold value $\mathrm{P}(\mathbf{r})-\gamma N_{t}^{0, \mathrm{~m}}$, while $\delta P$ is the amplitude of the $\cos (2 \mathrm{~m} \varphi)$ reservoir pumping harmonic, assuming that the coordinates are chosen so that the corresponding $\sin (2 \mathrm{~m} \varphi)$ harmonic has a zero amplitude. Note that the introduced pseudospin $\mathbf{s}$ is not related to the polarization degree of freedom of the exciton-polariton.

The evolution of the angular momentum pseudospin, obtained from Eq. (5), is then governed by the equation

$$
\begin{equation*}
\frac{d \mathbf{S}}{d \tau}=(\mathcal{P}-S) \mathbf{S}+\left(\delta \mathbf{P}-\mathbf{S}_{\|}\right) \frac{S}{2}+\left\{\left[\varepsilon \delta \mathbf{P}-(\xi-\varepsilon) \mathbf{S}_{\perp}\right] \times \mathbf{S}\right\} \tag{8}
\end{equation*}
$$

with introduced dimensionless values $\tau=t \gamma, \quad \mathbf{S}=$ $\beta \Gamma I_{c r} p_{t}^{0, \mathrm{~m}} \mathbf{s} / \gamma^{2}, \mathbf{S}_{\|}$and $\mathbf{S}_{\perp}$ being the projections of $\mathbf{S}$ onto the $x y$ plane and the $z$ axis respectively, $\mathcal{P}=\beta P / \gamma$, and $\delta \mathbf{P}=\beta \delta P / \gamma \mathbf{e}_{x}$ with $\mathbf{e}_{x}$ being the $x$ axis unitary vector. Taking into account that $\alpha_{1} \approx|X|^{2} \alpha$ with $X$ being the excitonic

Hopfield coefficient [42], the two interaction parameters read

$$
\begin{equation*}
\varepsilon=\frac{\alpha}{2 \beta}, \quad \xi=\frac{\alpha}{\beta} \frac{|X|^{2}}{p_{t}^{\mathrm{m}}} \frac{I_{c c}}{I_{c r}} \frac{\gamma}{\Gamma} . \tag{9}
\end{equation*}
$$

The first two bracketed terms of Eq. (8) represent gain-loss competition in the inhomogeneous pumping, while the last term describes absolute value conserving precession in the effective field. The latter in turn has two contributions, one of them stemming from the pumping asymmetry, quantified by $\delta P$, and the other one being the self-induced Larmor field [43] with the effective interaction prefactor $\xi-\varepsilon$. Depending on the relation between the effective polariton-exciton $(\varepsilon)$ and polariton-polariton $(\xi)$ interaction parameters the condensate is either in the repulsive $(\xi>\varepsilon)$ or in the attractive $(\xi<\varepsilon)$ regime.

On the other hand, by introducing $\phi_{ \pm}=\sqrt{2 \beta I_{c r}} \psi_{ \pm}$we obtain the nondimensionalized version of Eq. (5):

$$
\begin{align*}
i \frac{d \phi_{ \pm}}{d \tau}= & \left(\varepsilon+\frac{i}{2}\right)\left[\mathcal{P} \phi_{ \pm}+\frac{\delta \mathrm{P}}{2} \phi_{\mp}\right] \\
& +\left(\xi-\varepsilon-\frac{i}{2}\right)\left[\frac{1}{2}\left|\phi_{ \pm}\right|^{2}+\left|\phi_{\mp}\right|^{2}\right] \phi_{ \pm} \tag{10}
\end{align*}
$$

Again the physical meaning of the two terms is transparent. The first term with prefactor $\varepsilon+i / 2$ is linear in $\phi_{ \pm}$; it provides condensate gain and a non-Hermitian coupling of the two vortex states, stemming from the asymmetry of the pumping, governed by $\delta \mathrm{P}$. The second nonlinear term has a prefactor $\xi-\varepsilon-i / 2$, which stems from the fact that the polariton repulsion strength $\xi$ is partly compensated by reduction of repulsion off the reservoir through reservoir depletion. It also provides a pumping saturation stemming from angular inhomogeneous reservoir depletion.

Equation (8) has a pair of trivial stationary solutions, for which $S_{y}=S_{z}=0$ and $S_{x}= \pm S$ corresponding to a petal state [e.g., shown in the inset of Fig. 1(a)] where the two counter-rotating harmonics $\Psi^{0, \pm \mathrm{m}}$ are phase locked with 0 and $\pi$ phase shifts. The pumping power dependence of these trivial solutions reads $S^{ \pm}(\mathcal{P})=(2 \mathcal{P} \pm \delta \mathrm{P}) / 3$. There exists also another pair of stationary solutions, characterized by spontaneous parity symmetry-breaking and nonzero $S_{z}$ and $S_{y}$ pseudospin components. These two stationary pseudospins have the same absolute values and $S_{x}$ components, but the opposite signs of both $S_{y}$ and $S_{z}$, thus corresponding to the opposite directions of vorticity. The transition between the two types of solutions explains the spontaneous formation of pinned quantum vortices in optical traps [44]. For these parity symmetry breaking solutions the fixed pseudospin positions are given by

$$
\begin{align*}
& S_{y}=-\frac{\mathcal{P}-S}{\varepsilon \delta \mathrm{P}} S_{z}, \quad S_{x}=\frac{3 S-2 \mathcal{P} \mathcal{P}-S}{2 a} \frac{\varepsilon \delta \mathrm{P}}{\varepsilon \delta \mathrm{P}}+\frac{\varepsilon-\xi}{\varepsilon-} \\
& S_{z}^{2}=\frac{\varepsilon}{a} \frac{\delta \mathrm{P}^{2}}{2} \frac{S}{\mathcal{P}-S}-\frac{1}{2}\left(\frac{\varepsilon \delta \mathrm{P}}{a}\right)^{2} \frac{2 \mathcal{P}-3 S}{\mathcal{P}-S}-\left(\frac{2 \mathcal{P}-3 S}{2 a}\right)^{2} . \tag{11}
\end{align*}
$$

Equation (10) yields a pair of equations on the common and relative phases of the spinor $\left[\phi_{+}, \phi_{-}\right]^{T}$,

$$
\begin{equation*}
\Phi=\frac{1}{4 i} \ln \left(\frac{\phi_{+} \phi_{-}}{\phi_{+}^{*} \phi_{-}^{*}}\right), \phi=\frac{1}{4 i} \ln \left(\frac{\phi_{+}^{*} \phi_{-}}{\phi_{+} \phi_{-}^{*}}\right) . \tag{12}
\end{equation*}
$$

The time derivative of the common phase $\Phi$ yields the energy of a condensate at a fixed point state $\mathbf{S}$ :

$$
\begin{equation*}
\Omega=\varepsilon \mathcal{P}+\frac{3}{2} a S+\frac{\delta \mathrm{P}}{2}\left[\frac{\varepsilon S_{x} S+S_{y} S_{z} / 2}{S^{2}-S_{z}^{2}}\right] \tag{13}
\end{equation*}
$$

and equating the time derivative of the relative phase to zero we get the quadratic equation on the nontrivial state population $S(\mathcal{P})$ :

$$
\begin{align*}
& 2 \xi\left(\varepsilon^{2} \delta \mathrm{P}^{2}+\mathcal{P}^{2}\right)-\mathcal{P} S\left[4 \xi+\varepsilon+4 \varepsilon a^{2}\right] \\
& \quad+S^{2}\left[2 \xi+\varepsilon+4 \varepsilon a^{2}\right]=0 \tag{14}
\end{align*}
$$

which yields

$$
\begin{equation*}
S(\mathcal{P})=\frac{\mathcal{P}}{2}\left[1+\frac{2 \xi}{2 \xi+\varepsilon+4 \varepsilon a^{2}}\right] \pm \frac{\varepsilon}{2} \sqrt{D(\mathcal{P})} \tag{15}
\end{equation*}
$$

where

$$
\begin{equation*}
D(\mathcal{P})=\mathcal{P}^{2}\left(\frac{1+4 a^{2}}{2 \xi+\varepsilon+4 \varepsilon a^{2}}\right)^{2}-\frac{8 \xi \delta \mathrm{P}^{2}}{2 \xi+\varepsilon+4 \varepsilon a^{2}} \tag{16}
\end{equation*}
$$

The pair of nontrivial solutions (15) appears at a saddle-node bifurcation $\mathcal{P}_{S N}$ in the effective repulsion regime $(\xi>\varepsilon)$, or develops at a pitchfork bifurcation $\mathcal{P}_{P 2}$ from the trivial solution in the effective attraction regime $(\xi<\varepsilon)$. The saddlenode bifurcation point may be found from the condition $D\left(\mathcal{P}_{S N}\right)=0$, which yields

$$
\begin{equation*}
\mathcal{P}_{S N}=\delta \mathrm{P} \sqrt{8 \xi\left(2 \xi+\varepsilon+4 a^{2}\right)} /\left(1+\varepsilon a^{2}\right) \tag{17}
\end{equation*}
$$

The stability of a stationary solution is characterized by the Lyapunov exponents, which may be obtained from the Jacobian of the pseudospin evolution equation (8). The lower trivial branch $S^{-}(\mathcal{P})$ is unstable, while the higher $S^{+}(\mathcal{P})$ evolves either to a limit cycle at the AndronovHopf bifurcation point $\mathcal{P}_{A H}=5 \delta \mathrm{P} / 2$ or to a pair of symmetry-breaking fixed points at the pitchfork bifurcation $\mathcal{P}_{P 1}=\delta \mathrm{P}\left[12 \varepsilon\left(a+\sqrt{a^{2}+1 / 4}\right)-1 / 2\right]$ provided $\mathcal{P}_{P 1}<\mathcal{P}_{A H}$ with $a=\xi-\varepsilon$. The latter condition may be expressed as $4 \varepsilon\left(a+\sqrt{a^{2}+1 / 4}\right)<1$. Figure 2(a) shows the regions of two interaction parameters, corresponding to these two scenarios of the trivial solution evolution with increasing pumping power $\mathcal{P}$. The stable vortex solutions, on the contrary, appear at a saddle-node bifurcation $\mathcal{P}_{S N}$ in the condensate repulsion regime and at a pitchfork bifurcation $\mathcal{P}_{P 2}$ in the attraction regime.

Interestingly, both in the repulsive and attracting regimes there is a region of parameters where the only stable solution is a limit cycle. In the attraction regime $(\xi<\varepsilon)$, where the trivial solution evolves according to Andronov-Hopf scenario with increasing $\mathcal{P}$, this is a region of pumping powers between $\mathcal{P}_{A H}$ and the vortex pitchfork bifurcation, where the limit cycle loop closes back to a stationary point and transforms to the vortex. In the repulsion regime $(\xi>\varepsilon)$ the condition on the presence of such a region reads $\mathcal{P}_{A H}<\mathcal{P}_{S N}$. The range of pumping powers corresponding to the limit


FIG. 2. (a) Phase diagram of polariton condensate bifurcations. Areas of parameters corresponding to Andronov-Hopf (pitchfork) bifurcation type of the trivial stable solution branch $S^{+}(\mathcal{P})$ are shown in green (yellow) color. The dashed line separates the repulsion (above) and attraction (below) regimes. The hatched blue area corresponds to the regime where limit cycles are the only orbitally stable solutions. The unhatched blue area corresponds to bistability between the trivial and the symmetry breaking solution. (b) Stationary condensate population dependence on pumping power. Stable (unstable) fixed points are plotted with solid (dashed) lines. Trivial $S^{+}(\mathcal{P})$ and $S^{-}(\mathcal{P})$ and the symmetry-breaking solutions are plotted with red (upper straight), blue (lower straight), and black (curved) lines. The gray area highlights the instability range, and the frequency of limit cycles is plotted with the green dash-dotted line.
cycles reaches its maximum width at the separation between the repulsion and attraction regimes, where two competing nonlinear mechanisms compensate each other $(\xi=\varepsilon)$. The region of parameters, where $\mathcal{P}_{P 1}>\mathcal{P}_{A H}>\mathcal{P}_{S N}$, on the other hand, corresponds to a bistability between the limit cycle and vortex solutions at some range of pumping powers. These periodically evolving condensate states are discussed in detail in the following section.

## III. SPACE-TIME ORDERED PHASE

In the following we discuss the nature of the periodic limit cycles, which is the only orbitally stable solution of Eq. (5) in the intermediate range of pumping powers between the stability regions of the trivial petal state and the parity breaking vortical state, as shown in Fig. 2(b). This range is only present in the Andronov-Hopf bifurcation scenario and if $\mathcal{P}_{A H}<$ $\mathcal{P}_{S N}\left(\mathcal{P}_{P 2}\right)$ in the repulsion (attraction) regime. This condition is satisfied in a certain region of interaction parameters $\xi$ and $\varepsilon$, which is highlighted with hatching in Fig. 2(a). Note that orbitally stable limit cycles are also present in the unhatched blue area in Fig. 2(a), where bistability between the petals and vortical states exists, and generally in the symmetry-breaking state region of stability. However, in this case the competition between the limit cycles and the symmetry-breaking fixed points, mostly governed by the volumes of corresponding basins of attraction, complicates reliable realization of the space-time ordered state.

## A. Two-mode model

Linearization of Eq. (8) in the vicinity of the bifurcation $\mathcal{P}_{A H}$ yields elliptic precession in the $y z$ plane with the frequency $\omega_{0}=\delta \mathrm{P} \sqrt{\varepsilon(2 \xi-\varepsilon)}$. For $\mathcal{P}>\mathcal{P}_{A H}$ it transforms into


FIG. 3. Limit cycles of the condensate evolution. (a) Pseudospin trajectories for pumping power spanning the instability range. (b) Spectral and temporal (inset) dependence of the pseudospin projections in the anharmonic limit cycle regime, corresponding to the blue line in panel (a).
anharmonic periodic rotation of the pseudospin, characterized with frequency combs, shown in Fig. 3(b). The equation on the evolution of the pseudospin $\mathbf{S}(\tau)$ (8) has been numerically solved within the Runge-Kutta (fourth-order) method. The initial conditions were set to $\mathbf{S}(\tau=0)=(0,0,1)$. The parameters used are $\varepsilon=\xi=2.5, \delta \mathrm{P}=0.01$. There was taken the equidistantly spaced range of $4 \mathcal{P}$ values within the $0.025 \leqslant$ $\mathcal{P} \leqslant 0.1225$ domain in order to span the area of unstable limit cycles. Figure 3(a) shows the limit cycle trajectories in the pseudospin space, corresponding to pumping powers increasing within the instability range, and oscillations of the pseudospin projections, similar to the polarization pseudospin oscillations recently observed in the pulsed excitation regime [45]. The inverse period of anharmonic periodic rotation $T^{-1}$, calculated as the main harmonic frequency of the Fourier transform $\mathbf{S}\left(\tau^{-1}\right)$ of the Eq. (8) numerical solution $\mathbf{S}(\tau)$, decreases from $\omega_{0} / 2 \pi$ to zero, as shown in Fig. 2(b).

## B. GPE simulation

We have also performed direct numerical integration of Eqs. (1) and (2) and demonstrated the destabilization of a petal state (wave-function harmonic of zero net angular momenta) and its subsequent evolution into a giant quantized vortex state (see Fig. 4). This mechanism corresponds either to a saddle-node $\left(\mathcal{P}_{S N}\right)$ or a pitchfork ( $\mathcal{P}_{P 2}$ ) bifurcation scenario depending on whether one is in the repulsive $(\xi>\epsilon)$ or attractive $(\xi<\epsilon)$ regime respectively. For Fig. 4 we have $\alpha / \beta=1.5, \gamma / \Gamma=10$, and a pump shape,

$$
\begin{equation*}
\mathrm{P}(\mathbf{r})=\mathrm{P}_{0} \exp \left[\frac{r-r_{0}}{w}\right]^{6} \tag{18}
\end{equation*}
$$

where $r=\sqrt{x^{2}+(y / 1.1)^{2}}, w=3 \mu \mathrm{~m}$ and $r_{0}=8 \mu \mathrm{~m}$. The factor 1.1 introduces geometric ellipticity to the pump shape which breaks the twofold degeneracy of the standing-wave (petal) states and allows stable petal state formation [35] [see Figs. 4(a) and 4(d)].

In order to determine whether we are in the attractive or repulsive regime we use Eq. (9). For a $\mathrm{m}=10$ harmonic it was estimated that $2 I_{c c} /\left(p_{t}^{0, \mathrm{~m}} I_{c r}\right) \sim 10$. In Fig. 4 we show a $\mathrm{m}=5$ harmonic which will have a lower overlap with the reservoir $\left(I_{c r}\right)$ because of a smaller centrifugal force, stronger


FIG. 4. (a)-(c) Spatial density profiles of an elliptically pumped polariton condensate and (d)-(f) corresponding phase profiles. In (a) and (d) the pump power is $5 \%$ above threshold and a stable petal pattern forms due to the geometric ellipticity of the pump. At $10 \%$ above threshold (b),(e) the petal pattern bifurcates (loses stability) and evolves into one of two counter-rotating vortex states (c),(f) with equal probability. The slight modulation in the density and finite distance between phase singularities in the vortex center (c),(f) stems from the elliptic profile of the pump.
overlap with itself $\left(I_{c c}\right)$ due to higher localization, and a lower threshold pump intensity $p_{t}^{0, \mathrm{~m}}$ [see Fig. 1(a)]. This tells us that $2 I_{c c} /\left(p_{t}^{0, \mathrm{~m}} I_{c r}\right)>10$ for lower modes. Using an exciton Hopfield coefficient of $|X|^{2}=0.01$, straightforward calculation gives $\epsilon=0.75$ and $\xi=3$ which tells us that the current results in Fig. 4 are in the repulsive regime and that a saddle-node bifurcation is taking place in Figs. 4(b) and 4(e).

We have demonstrated periodic evolution of the condensate, corresponding to limit cycles in the two-mode model, with direct numerical simulation. The condensate density time shots, obtained from the full numerical solution of GPE (1) and demonstrating its periodic evolution, are shown in Fig. 5. Numerics were performed using spectral methods in space and a variable-step, variable-order Adams-Bashforth-Moulton solver. In order to avoid gain at the boundaries we use ad hoc


FIG. 5. Numerical solution of GPE in the time crystal regime. (a) Polariton density time shot. (b) Evolution of instantaneous polariton density and current (shown with arrows) throughout the period $T$.
finite pump shape (18). The parameters were set to $\hbar m_{p}=$ $5 \times 10^{-5} m_{0}$ where $m_{0}$ is free electron mass, $\Gamma=0.05 \mathrm{ps}^{-1}$, $\gamma / \Gamma=0.05, \beta=5 \times 10^{-4} \mathrm{ps}^{-1} \mu \mathrm{~m}^{-2}, \alpha / \beta=1.6, \alpha_{1} / \beta=$ $6, P_{0} / \beta=2000, r_{0}=25 \mu \mathrm{~m}$, and $w=6 \mu \mathrm{~m}$. The wave function is initially seeded by stochastic white noise to mimic the incoherent uncondensed state.

## IV. DISCUSSION AND CONCLUSION

The experimental conditions for the realization of the space-time crystal regime are twofold. The spatial order is inherent for the linear limit of the GPE (1) and requires $R \sqrt{2 m_{p} \Gamma}>1$ so that $\mathrm{m}>1$, as follows from Fig. 1(b). The temporal order in turn emerges in the nonlinear regime above the condensation threshold provided $\xi \approx \varepsilon$ [see Fig. 2(a)]. Estimating $2 I_{c c} /\left(p_{t}^{0, \mathrm{~m}} I_{c r}\right) \sim 10$ for $\mathrm{m} \sim 10$, and $\gamma / \Gamma \sim 10[46,47]$, the condition on the interaction parameters transforms to the condition on the Hopfield coefficient $|X|^{2} \sim 0.01$.

In contrast to existing realizations of space-time crystals, the periodicity of the condensate oscillations is governed by the optical trap parameters rather than the optical pumping frequency. The inverse oscillation period scales from $\alpha \delta P / \hbar \gamma$ to zero while the pumping power spans the limit cycle instability range [see Fig. 2(b)], suggesting that the time crystal regime may be observed with time resolution of polariton emission by fine tuning the optical trap parameters.

The mean-field model studied in this work is valid above the polariton lasing threshold, where the condensate occupation number is macroscopic. One should note, however, that the fluctuations of the occupation, supplemented with polariton-polariton interactions, result in the condensate wave-function decoherence [48]. The temporal ordering of the condensate evolution is thus limited with the coherence time, which is typically long compared to polariton lifetime, and the spatial ordering is in turn limited by the coherence length, which by far exceeds the characteristic optical trap size [12].

The oscillating character of the condensate evolution is reminiscent of the Josephson oscillations of coupled BoseEinstein condensates [49,50]. However, in contrast to condensates with conserved number of particles, where the exact form of wave function evolution depends on the initial conditions, here the dynamically stable oscillations of the open polariton system are only governed by the optical trap parameters. This limit cycle motion is strongly anharmonic, as is seen from the inset of Fig. 3(b). One should also note that the spatial ordering, which is present in trapped high angular momentum polariton condensates, is an important prerequisite of time crystallization [5].

In conclusion, we predict a space-time ordered phase of polariton condensates, created and trapped by excitonic reservoirs of annular shapes, which has the properties of a time crystal. This phase arises from limit cycle instability in the vicinity of spontaneous parity breaking transition from petals to quantum vortices and can be described in terms of the condensate pseudospin rotation. The physical origin of the emerging space-time order is in the interplay of strong interactions and driven-dissipative nature of exciton-polariton condensates.

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