Supplemental material: Hyperbolic Metamaterials with Bragg Polaritons

I. BRAGG MIRROR MODEL. EIGENMODE EQUATION

We consider a semiconductor Bragg mirror schematically shown in Fig. 1. The structure consists of the periodic array of the alternating dielectric layers with quantum wells (QWs) placed in the centres of one type of the layers. To

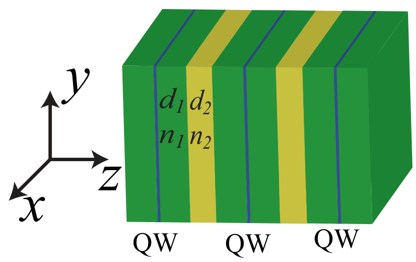

FIG. 1. Schematic image of the structure. Infinite Bragg mirror infiltrated with quantum wells.

obtain the dispersion equation for the eigenmodes of this structure we use the transfer matrix technique[1]. A general dispersion equation for the periodic structure can be written as

$$\cos(KD) = \frac{1}{2}\mathrm{Tr}(\hat{T}),$$

where K is the Bloch wavevector, D is the period of the structure, and $\hat{T}$ is the transfer matrix across the period of the structure. In the case of our structure, $\hat{T}$ can be written as a matrix product

$$\hat{T} = \hat{T}_{d_1/2}\hat{T}_{QW}\hat{T}_{d_1/2}\hat{T}_{d_2},$$

where $\hat{T}_{d_1/2}$ is the transfer matrix for the propagation over the half of the first layer. Further on we assume the s-polarization of light (the electric field has no component, perpendicular to the layer interfaces). In this case the transfer matrix is given by

$$\hat{T}_{d_1/2} = \begin{pmatrix} \cos(k_{z1}d_1/2) & \frac{ik_0}{k_{z1}}\sin(k_{z1}d_1/2) \\ \frac{ik_{z1}}{k_0}\sin(k_{z1}d_1/2) & \cos(k_{z1}d_1/2) \end{pmatrix},$$

where $k_0 = \omega/c$, and $k_{z1} = \sqrt{n_1^2k_0^2 - \mathbf{k}_\rho^2}$, and $\mathbf{k}_\rho = (k_x, k_y)$ is the in-plane wavevector component; n_1 is the refractive index of the first layer. The transfer matrix for the second layer is analogous.

The transfer matrix from the exciton QW can be written in the form

$$\hat{T}_{QW} = \begin{pmatrix} 1 & 0 \\ 2\frac{k_z r}{k_0 t} & 1 \end{pmatrix},$$

where r and t are the reflection and transmission coefficients for the QW which in the case of the s-polarization are given by

$$r = \frac{in_1k_0\Gamma_0/k_{z1}}{\omega_X' - \omega - i(\Gamma + n_1k_0\Gamma_0/k_{z1})}, \tag{1}$$

$$t = 1 + r, \tag{2}$$

where Γ_0 is the exciton radiative decay rate, ω_X' is the renormalized exciton frequency which in the approximation of infinitely thin QWs could be set equal to the exciton frequency ω_X, and Γ is the nonradiative exciton decay rate. Finally, we obtain the following dispersion equation:

$$\begin{aligned}\cos(KD) = {} & \cos(k_{z1}d_1)\cos(k_{z2}d_2) - \frac{1}{2}\sin(k_{z1}d_1)\sin(k_{z2}d_2)\left(\frac{k_{z1}}{k_{z2}} + \frac{k_{z2}}{k_{z1}}\right) \\ & + \frac{ir}{t}\left(\sin(k_{z1}d_1)\cos(k_{z2}d_2) + \sin(k_{z2}d_2)\left[\frac{k_{z1}}{k_{z2}}\cos^2(k_{z1}d_1/2) - \frac{k_{z2}}{k_{z1}}\sin^2(k_{z1}d_1/2)\right]\right).\end{aligned} \tag{3}$$

This equation implicitly defines the polariton eigenfrequency $\omega(K, k_x, k_y)$. In the absence of losses, for the fixed value of K and k_ρ Eq. (3) has infinite number of solutions, corresponding to the infinite number of photonic bands in photonic crystal. We however focus at the four solutions, corresponding to the coupling of exciton to two photonic bands which have band centre frequencies closest to the exciton resonance.

In our calculations, we have considered the GaN/$Al_{0.3}Ga_{0.7}N$ Bragg mirror with thin $In_{0.12}Ga_{0.88}N$ QWs in the centres of the GaN layers as a model system. The second photonic band gap centre $\hbar\omega_B$ was tuned to 3 eV. The refractive indices of the layers are $n_1 = n_{\mathrm{GaN}} = 2.55$, $n_2 = n_{\mathrm{AlGaN}} = 2.15$ and the thicknesses of the layers $d_1 = 64.8$ nm, $d_2 = 115.3$ nm. The radiative decay rate $\hbar\Gamma_0$ of the InGaN quantum well exciton is 2 meV and the nonradiative rate $\hbar\Gamma$ is 0.1 meV. The exciton energy $\hbar\omega_X$ is tuned to 2.95 eV. The dispersions of the eigenmodes for the given parameters defined by Eq. (3) are shown in Fig. 2.

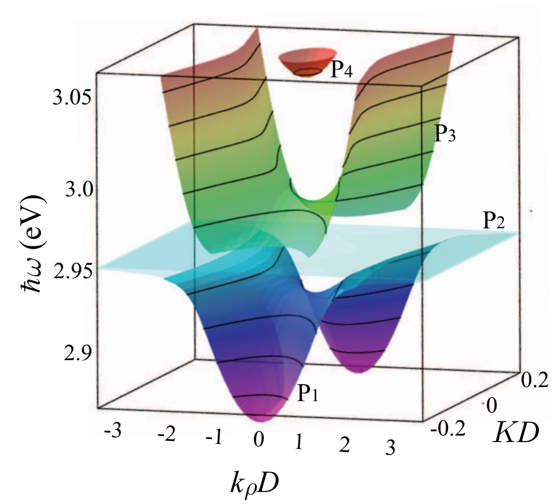

FIG. 2. Dispersion surface for the GaN/$Al_{0.3}Ga_{0.7}N$ Bragg mirror with $In_{0.12}Ga_{0.88}N$ QWs placed in the centres of the GaN layers. Refractive indices of the layers: $n_{\mathrm{GaN}} = n_1 = 2.55$, $n_{\mathrm{AlGaN}} = n_2 = 2.15$, refractive index contrast of QWs is neglected. The thicknesses of the layers are $d_1 = 64.8$ nm, $d_2 = 115.3$ nm. The QW exciton frequency is tuned to the upper band edge of the second photonic band, $\hbar\omega_X = 2.95$ eV.

In order to obtain the analytical approximations for the eigenenergy and effective mass tensor of the lower polariton branch we first derive the band gap width. We take into account, that the even photonic band gaps are degenerate in the quarter-wavelength Bragg mirror, where $n_1 d_1 = n_2 d_2$. Introducing the parameter $\xi = n_1 d_1 / n_2 d_2$ and setting $K = 0, \mathbf{k}_\rho = 0, \Gamma_0 = 0$ we then expand Eq. (3) over the small parameters $(\xi - 1)$ and $\delta = \omega/\omega_B - 1$, where $\omega_B = 2\pi c/(n_1 d_1 + n_2 d_2)$ is the centre of the second photonic band gap. Expansion up to the second order with respect to δ and $(\xi - 1)$ gives us the second order algebraic equation for δ which has the solutions

$$\tilde{\delta} = \pm \frac{1}{2} \frac{(n_2 - n_1)(1 - \xi)}{(n_1 + n_2)},$$

which correspond to the value of the half-width of the band gap $\Omega_B = \omega_B |\tilde{\delta}| \approx 2\pi \times 12.1$ THz.

In the absence of QWs two resulting photonic bands have different symmetry properties. Namely, the electric field of the lower band in the Γ point is symmetric in each layer and for the upper band it is antisymmetric at the band edge. Thus, in the vicinity of the Γ point the upper band is not coupled to the excitonic mode and there are almost bare flat excitonic mode and photonic mode which can be seen in Fig. 2.

Now we concentrate on the lower photonic mode. To obtain the analytical approximations for the effective masses we have to expand Eq. (3) for $K, \mathbf{k}_\rho \ll 1/D$, and for $\Gamma_0 = 0$ and for the frequencies in the vicinity of the band edge $\omega \approx \omega_B - \Omega_B$. We also should perform the expansion over small $\xi - 1$. We finally obtain the expression for the photonic branch dispersion:

$$\omega_{ph1} = \omega_B - \Omega_B + \frac{1}{2} \frac{(n_1^2 - n_1 n_2 + n_2^2)}{n_1^2 n_2^2} \frac{c^2}{\omega_B - \Omega_B} k_\rho^2 - \frac{1}{2} \frac{n_1 n_2}{\pi^2 (n_1 + n_2)^2} \frac{(\omega_B - \Omega_B)\omega_B}{\Omega_B} D^2 K^2.$$

From this equation we can obtain the effective mass tensor components

$$m_{ph\perp} = -\frac{\hbar\Omega_B}{(\omega_B-\Omega_B)\omega_B D^2}\frac{\pi^2(n_1+n_2)^2}{n_1 n_2} \approx -0.4\times 10^{-33}\mathrm{g}, \tag{4}$$

$$m_{ph\parallel} = \frac{\hbar(\omega_B-\Omega_B)}{c^2}\frac{n_1^2 n_2^2}{(n_1^2 - n_1 n_2 + n_2^2)} \approx 2.8\times 10^{-32}\mathrm{g}. \tag{5}$$

Finally, in order to obtain the polariton dispersion, we first set $K = 0, \mathbf{k}_\rho = 0$ and expand the Eq. (3) in the vicinity of $\omega = \omega_B - \Omega_B$ but now for $\Gamma_0 \neq 0$ and for $\omega_0' = \omega_B - \Omega_B$. As a result, for the value of the Rabi splitting Ω_P we obtain:

$$\Omega_P = \sqrt{\frac{n_2}{(n_1+n_2)\pi}\Gamma_0(\omega_b - \Omega_B)}.$$

To obtain the effective mass of lower polariton branch we have to expand Eq. (3) in the vicinity of $\omega = \omega_b - \Omega_B - \Omega_P$ for the small K, k_ρ which gives the conventional result for the low-branch polariton masses in zero detuning case[2]:

$$m_\perp = 2m_{ph\perp} \approx -0.8\times 10^{-33}\mathrm{g}, \tag{6}$$

$$m_\parallel = 2m_{ph\parallel} \approx 5.6\times 10^{-32}\mathrm{g}. \tag{7}$$

In Fig. 3 we show that the obtained analytical expressions are in good agreement with the numerical dispersion relations in the vicinity of the band edge. Thus, the effective mass approximation in the region $|KD| < 0.1$ and $|k_\rho D| < 0.5$ is fulfilled.

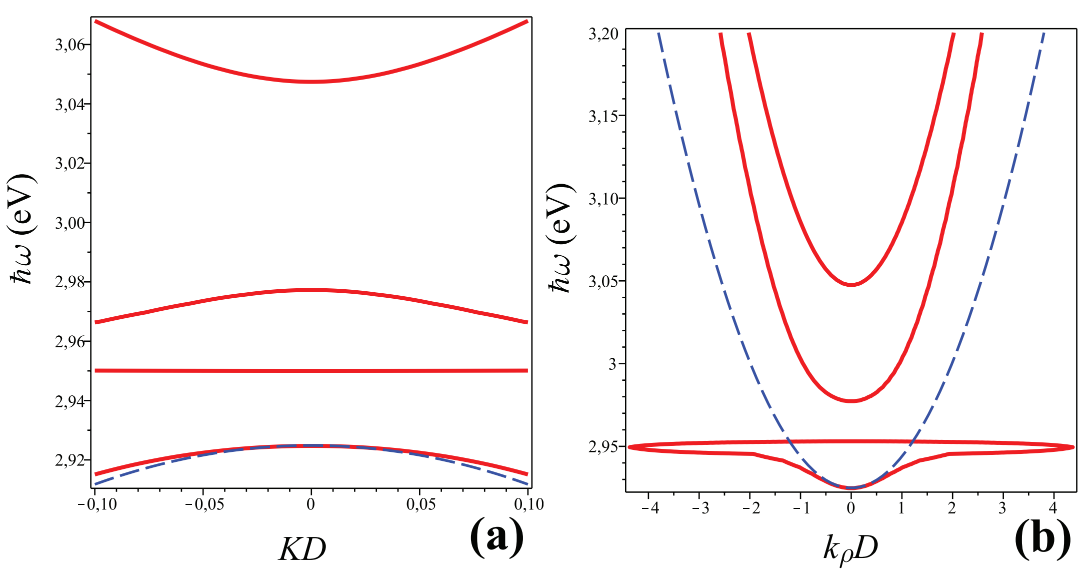

FIG. 3. Dispersion characteristics calculated numerically solving Eg. (3) (solid red lines) and by using effective mass approximation (dashed blue lines) for the lower polariton branch for (a) – $k_\rho = 0$ and (b) – $K = 0$.

II. GROSS-PITAEVSKII EQUATION FOR BRAGG POLARITONS

We consider only the lowest polariton branch and derive the Gross-Pitaevski equation for the polariton coherent state introducing the kinetic energy operator

$$\hat{H}_0 = \hbar\omega(i\hbar\frac{\partial}{\partial z}, i\hbar\frac{\partial}{\partial x}, i\hbar\frac{\partial}{\partial y}).$$

The analytical expression for the kinetic energy operator is available only within the effective mass approximation, which holds for $k_\rho, K \ll \pi/D$ and yields

$$\hat{H}_0 = \hbar\left(\omega_B - \Omega_B - \Omega_P\right) + \frac{\hbar^2 K^2}{2m_\perp} + \frac{\hbar^2 k_\rho^2}{2m_\parallel}. \tag{8}$$

Thereafter we will omit the constant energy $E_0 = \hbar\left(\omega_B - \Omega_B - \Omega_P\right)$. We now obtain the Gross-Pitaevskii equation for the lover polariton branch $\mathcal{P}_1$ without any loss of generality. The nonlinear term in the Hamiltonian written in k-space is given by

$$\begin{aligned} V_{nl} &= \frac{6E_b\; a_b^3 X_1^4 (D/d_{QW})}{V} \sum_{k_1,k_2,q} \mathcal{P}^\dagger_{1,k_1+q} \mathcal{P}^\dagger_{1,k_2-q}, \mathcal{P}_{1,k_1} \mathcal{P}_{1,k_2} \\ &= \frac{g_0}{2} X_1^4 \sum_{k_1,k_2,q} \mathcal{P}^\dagger_{1,k_1+q} \mathcal{P}^\dagger_{1,k_2-q} \mathcal{P}_{1,k_1} \mathcal{P}_{1,k_2}, \end{aligned} \tag{9}$$

where X_1 is the excitonic Hopfields coefficient for the lowest polariton branch which in the case of zero-detuning and $KD, k_\rho D \ll 1$ can be approximated as $X_1 \approx 1/\sqrt{2}$. Then, if in the reciprocal space nonlinear potential is wavevector independent, in the real space the potential may be approximated by a delta-function $V_{nl}(|r - r'|) = g\delta(r - r')$, where $g = g_0 V X_1^4$.

Then we introduce the bosonic field operator

$$\hat{\Psi} \equiv \hat{\Psi}_{\mathcal{P}_1}(r,t) = \frac{1}{\sqrt{V}} \sum_k \mathcal{P}_1(k) e^{ikr - i\omega_k t} \tag{10}$$

that obeys the commutation relations:

$$[\hat{\Psi}(r), \hat{\Psi}^\dagger(r')] = \delta(r - r'), \tag{11}$$

and all other commutators are zero. The Hamiltonian of the system in the real space reads

$$\hat{H} = \int d^3r \hat{\Psi}^\dagger(r) \hat{H}_0 \hat{\Psi}(r) + \frac{g}{2} \int d^3r \hat{\Psi}^\dagger(r) \hat{\Psi}^\dagger(r) \hat{\Psi}(r) \hat{\Psi}(r). \tag{12}$$

We then apply the commutation relations to obtain

$$i\hbar \frac{\partial \hat{\Psi}}{\partial t} = \hat{H}_0 \hat{\Psi} + g \hat{\Psi}^\dagger \hat{\Psi} \hat{\Psi}. \tag{13}$$

Finally, we assume that there is a macroscopic occupation in the ground state of $\mathcal{P}_1$. Next, we use mean-field approach to replace the corresponding polariton field operator $\hat{\Psi}(\mathbf{r})$ by its mean value $\langle \hat{\Psi}(\mathbf{r}) \rangle \equiv \Psi(\mathbf{r})$, which characterizes the low branch polariton wave function[3]. We obtain a governing Gross-Pitaevskii equation for $\Psi(\mathbf{r})$

$$i\hbar \frac{\partial \Psi}{\partial t} = \left[-\frac{\hbar^2}{2m_\parallel} \Delta_\parallel - \frac{\hbar^2}{2m_\perp} \frac{\partial^2}{\partial z^2} + g|\Psi|^2 \right] \Psi.$$

[1] L.M. Brekhovskikh, *Waves in Layered Media* (Academic, New York, 1980).
[2] A.V. Kavokin, J. Baumber, G. Malpuech, F. Laussy, *Microcavities* (Oxford University Press, 2007).
[3] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999).